TAKING WITTGENSTEIN AT HIS WORD

A TEXTUAL STUDY

by

Robert J. Fogelin

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PRINCETON UNIVERSITY PRESS

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For Walter Sinnott-Armstrong

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THE DIFFICULT THING HERE IS NOT, TO DIG
DOWN TO THE GROUND; NO, IT IS TO RECOGNIZE
THE GROUND THAT LIES BEFORE US
AS THE GROUND.

—*RFM* VI 31

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Preface

A few words about the aim of this work may help explain the form it takes. As I will spell out more fully in the Introduction, this work is a textual study organized around a central question: What kind of interpretation emerges if we adhere strictly to Wittgenstein's methodological pronouncements, in particular, his claims that his aim is purely therapeutic and that he is not in the business of presenting and defending philosophical theses? This work is an experiment in reading a selection of central themes in Wittgenstein's later philosophy under the constraints of Wittgenstein's own methodological pronouncements.

Since this is a textual study, I largely let Wittgenstein speak for himself. The task of interpreting Wittgenstein's later writings is not—as with his *Tractatus*—one of deciphering opaque passages. Sentences in his later writings are usually transparent as they stand. The challenge, where there is one, is to appreciate the philosophical significance that Wittgenstein assigns to them. In this regard, Wittgenstein is his own best commentator. His self-commentaries take various forms. Some appear as aphorisms concerning the proper way of approaching philosophical

problems. Others involve internal dialogues—usually with an imagined interlocutor, sometimes with himself—where pathways into philosophical confusion are acted out. In this way Wittgenstein shows how philosophical confusions can naturally arise and seem to be forced on us when we are engaged in the detached setting of philosophical reflection.

Once, in a conversation I had with Brian McGuiness, he compared some Wittgenstein scholars with those who attempt to explain Homer by citing Homer. That is precisely the sort of thing I am doing by taking Wittgenstein as his own best commentator. This explains why I say relatively little about the secondary literature on Wittgenstein's later philosophy, my own writing not exempted. I do not want to get deeply involved with the secondary literature because such a discussion, if it is to be done in a fair and systematic manner, will inevitably lead away from the text into a complex taxonomical examination of competing interpretations. Was Wittgenstein a behaviorist, a finitist, a contextualist, a coherentist, an anti-realist, a pragmatist, a verificationist, a Whorfian, a conventionalist, a nominalist, a constructivist, a skeptic, a nihilist, and so on? Modest comparisons of Wittgenstein's philosophical strategies with positions held by others can be useful. But there is a tendency to push things further. With Wittgenstein's philosophy fixed within the grid of possible positions, we can then ask: What *version* of the position does he adopt? Does he consistently adhere to it? Wittgenstein aside (!), is the position he adopts independently justifiable? Philosophical "geography" of this kind can be useful—and some people are very good at it—but it is alien to Wittgenstein's manner of doing philosophy and I will not adopt it.

This work does not present a novel—hitherto unrec-

ognized—way of understanding Wittgenstein's later philosophy. It contains a fair amount that will, for Wittgenstein scholars, seem old hat. To some extent, I am simply "assembling reminders" of Wittgenstein's conception of the proper way of doing philosophy and then showing how, in particular cases, his investigations conform to it. Sometimes this amounts to saying, "Look at this!"

On a number of occasions, Wittgenstein expressed exasperation concerning the fate of his philosophical reflections. As he put it in the preface in *Philosophical Investigations*:

> I [learned] that my results (which I had communicated in lectures, typescripts and discussions), variously misunderstood, more or less mangled or watered down, were in circulation.

He seemed to despair of correcting these misunderstandings:

> It is not impossible that it should fall to the lot of this work, in its poverty and in the darkness of this time, to bring light into one brain or another—but, of course, it is not likely.

Surveying the contemporary philosophical scene, one sometimes gets the impression that Wittgenstein's influence survives, to the extent that it does, only as empty slogans of a bygone era. Wittgenstein seemed to see this coming. He closed his 1939 lectures on the foundations of mathematics with these words:

> The seed I'm most likely to sow is a certain jargon.

Wittgenstein, in fact, finds himself in a peculiar position. Given his own diagnosis of philosophical misunderstandings and his appreciation of their deep entrench-

ment, it would seem inevitable that his own reflections would be transformed into philosophical theses representing one sort of philosophical standpoint or another—the very thing he warns us against. This work is an attempt to take his warnings seriously and, in the process, exhibit the richness and originality of Wittgenstein's manner of approaching philosophical issues.

These remarks are not intended to slight the work of other interpreters of Wittgenstein's philosophy. Much of it is first-rate and, over the years, important to me. The magisterial commentaries by G. P. Baker and P.M.S. Hacker are indispensable for anyone working in the field of Wittgenstein scholarship. I have found Garth Hallett's commentary very useful as well. I hold the writings of Brian McGuiness in high regard and regret he has yet to bring his biographical-philosophical account of Wittgenstein's life and works to completion. David Stern has important things to say about Wittgenstein's use of inner dialogues. Four decades ago, Carl Posey, who attended one of my graduate seminars, set me straight on various matters concerning the philosophy of mathematics. Both in their writings and in conversations, I have found the views of Barry Stroud and Hans Sluga insightful and simpatico. Both recognize the depth and importance of Wittgenstein's critique of the philosophical enterprise. On occasion, Sluga outdoes me. Students who studied Wittgenstein with me have played an important role in shaping my views. Many of their names have not survived my one-term memory, so I will not attempt to list them.

I have received institutional support for this project from the Rockefeller Study Center at Bellagio, the Liguria Study Center for the Arts and Humanities at Bogliasco, the Faculty Research Fund at Dartmouth College,

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I would also like to thank Harry Frankfurt, Rob Tempio, and Jodi Beder at the Princeton University Press for their encouragement and editorial help. Once more I wish to express my gratitude to Florence Fogelin and Jane Taylor for bringing their sharp eyes, intelligence, and sense of style to bear on drafts of this work.

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Conventions for Citations and Abbreviations

Citations to Wittgenstein's works will be given in the text; all other citations will be given in footnotes. References to Wittgenstein's text will employ the following abbreviations:

The Blue and Brown Books (*B & B*)

Lectures on the Foundations of Mathematics, Cambridge, 1939 (*LFM*)

Notebooks 1914–1916 (*NB*)

On Certainty (*OC*)

Philosophical Grammar (*PG*)

Philosophical Investigations (*PI*)

Philosophical Remarks (*PR*)

Remarks on the Foundations of Mathematics (*RFM*)

Remarks on the Philosophy of Psychology (*RPP*)

Tractatus Logico-Philosophicus (*TLP*)

Wittgenstein's Lectures: Cambridge, 1932–1935 (*WLC*)

Zettel (*Z*)

Where works by Wittgenstein are divided into parts and sections, Roman numerals will be used for parts and Arabic numerals for sections, with no comma between them. For example:

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PI 43 (Citations to the first part of *Philosophical Investigations* do not give the part number. References to part 2 have the form *PI* II, p. 200.)

Where there are no section numbers, page numbers are used instead: (*LFM*, p. 123). References to the *Tractatus* use Wittgenstein's numbering system: (*TLP* 2.12).

References to works by other writers refer the reader to the bibliography. Where the work cited appears in a number of editions, additional bibliographical information is sometimes given.

Throughout this work, I have respected both the punctuation and spelling in texts I have cited.

Introduction

Respecting the Text

In *Wittgenstein* (1976 and 1987) I located what I took to be the key move in Wittgenstein's reflections on the possibility of a private language in *PI* 198. There Wittgenstein presents the following problem concerning rule-following:

> *PI* 198. "But how can a rule shew me what I have to do at this point? Whatever I do is, on some interpretation, in accord with the rule."—That is not what we ought to say, but rather: any interpretation still hangs in the air along with what it interprets, and cannot give it any support. Interpretations by themselves do not determine meaning.

Three sections later, he refers to this problem as a paradox:

> *PI* 201. This was our paradox: no course of action could be determined by a rule, because every course of action can be made out to accord with the rule. The answer was: if everything can be made out to accord with the rule, then it can also be made out to conflict with it.

> And so there would be neither accord nor conflict here.

On the basis of this paradox, Wittgenstein then presents the following strong anti-privacy claim concerning rule-following:

> *PI* 202. And hence also 'obeying a rule' is a practice. And to think one is obeying a rule is not to obey a rule. Hence it is not possible to obey a rule 'privately': otherwise thinking one was obeying a rule would be the same thing as obeying it.

From earlier sections of *Philosophical Investigations* we know that Wittgenstein holds:

> (1) Language is a rule-governed activity.

So when, in *PI* 202, he explicitly tells us that

> (2) It is not possible to obey a rule 'privately,'

the thesis that a private language is impossible seems to follow at once.

This is not an implausible reading of the text, for it squares with much else that Wittgenstein says on the possibility of a private language. All the same, I now find this reading out of focus. It runs counter to Wittgenstein's claim—his insistence—that he is not engaged in presenting and defending philosophical theses. I have in mind passages of the following kind:

> *PI* 109. It was true to say that our considerations could not be scientific ones. It was not of any possible interest to us to find out empirically "that, contrary to our preconceived ideas, it is possible to think such-and-such"—whatever that may mean. . . . And we may not

> advance any kind of theory. There must not be anything hypothetical in our considerations. We must do away with all *explanation*, and description alone must take its place. And this description gets its light, that is to say its purpose, from the philosophical problems. These are, of course, not empirical problems; they are solved, rather, by looking into the workings of our language, and that in such a way as to make us recognize those workings: *in despite of* an urge to misunderstand them. The problems are solved, not by giving new information, but by arranging what we have always known. Philosophy is a battle against the bewitchment of our intelligence by means of language.

> *PI* 124. Philosophy may in no way interfere with the actual use of language; it can in the end only describe it.
>
> For it cannot give it any foundation either.
>
> It leaves everything as it is.

> *PI* 126. Philosophy simply puts everything before us, and neither explains nor deduces anything.—Since everything lies open to view there is nothing to explain. For what is hidden, for example, is of no interest to us.
>
> One might also give the name "philosophy" to what is possible *before* all new discoveries and inventions.

Wittgenstein's methodological passages have, of course, been noticed and remarked on by most commentators. But they are not always accepted as fundamental guides to understanding Wittgenstein's reflections on specific topics. After a brief (perhaps solemn) acknowledgment, they often drift out of sight. Crispin Wright, however, with admirable forthrightness, explicitly pushes these methodological pronouncements aside in these words:

> [I]t is difficult to reconcile Wittgenstein's pronouncements about the kind of thing which he thinks he ought to be doing with what he actually seems to do. Not that his actual treatment of the particular issues seems flatly inconsistent with his general methodological ideas. Rather, we can put the would-be interpreter's difficulty like this: it is doubtful how anyone who read only a bowdlerized edition of the *Investigations*, from which all reference to philosophical method and the nature and place of philosophy had been removed, would be able to arrive at the conclusion that the author viewed those matters in just the way in which Wittgenstein professes to do. At the time I write this, the complaint is justified that the great volume of commentary on the *Investigations* has so far done very little to clarify either how we should interpret the general remarks on philosophy so as to have our understanding enhanced of Wittgenstein's treatment of specific questions, or conversely. (What are the 'well-known facts' arranged in the course of the Private Language discussion?) Wittgenstein's later views on philosophy constitute one of the so far least well understood aspects of his thought. (Wright 1980, p. 262)

Michael Dummett also rejects some of Wittgenstein's central constraints—in particular, his descriptivism—but does not suggest, as Wright seems to, that Wittgenstein's methodological restrictions are mere hand-waving.

> We all stand, or should stand, in the shadow of Wittgenstein, in the same way that much earlier generations once stood in the shadow of Kant. Some things in his philosophy, however, I cannot see any reason for accepting: and one is the belief that philosophy, as such, must never criticize but only describe. This belief

> was fundamental in the sense that it determined the whole manner in which, in his later writings, he discussed philosophical problems; not sharing it, I could not respect his work as I do if I regarded his arguments and insights as depending on the truth of that belief. (Dummett 1991, p. xi)

Perhaps an interesting philosophical position with a Wittgensteinian cast could emerge with Wittgenstein's methodological restrictions relaxed. That, however, is not my present concern.

As indicated in the preface, I will not get deeply involved with the secondary literature on Wittgenstein's philosophy. There are, however, some exceptions. For my purposes it important to distinguish my reading of Wittgenstein's so-called skeptical paradox of rule-following found in *PI* 201 from one put forward some years later by Saul Kripke. The matter at issue is not one of priority. I do not claim to have published a Kripke-like interpretation of *PI* 201 four years before he did. My claim is that Kripke's understanding of the paradox in *PI* 201 is fundamentally wrong and I want to take precautions against having criticisms that have been leveled against his understanding of the paradox also leveled against mine. I will go into all this at the beginning of the next chapter. In chapter 2, I adopt a treatment of the notion of a *criterion* that is significantly different from one forcefully presented by P.M.S. Hacker in *Meaning and Mind* (1990). There are other references to the secondary literature—mostly acknowledgments. In general, however, I stay within the margins of Wittgenstein's text.

Returning to *PI* 109, it can be taken as a representative of the kind of passage that Crispin (the Expunger) Wright would delete. In some ways it may seem perplex-

ing. It begins by declaring that "it was true to say that our considerations could not be scientific ones." This is a view that Wittgenstein expressed in the *Tractatus*, and perhaps he is alluding to it here. (See, for example, *TLP* 4.111.) He then makes the more specific point that he is not engaged in an empirical investigation intended to establish new facts—facts that may be contrary to "our preconceived ideas." A few sentences later, he tells us that the philosophical problems that concern him "are, of course, not empirical problems; they are solved, rather, by looking into the workings of our language, and that in such a way as to make us recognize those workings." This may seem strange, for isn't "looking into the workings of our language" itself an empirical investigation? If so, in the same paragraph, Wittgenstein seems both to renounce and to recommend an empirical investigation of language. What is going on? The brief answer is this. Wittgenstein is not using the notion of an empirical investigation in a wide sense where the perceptual examination of any object—say, a spot on one's tie—counts as an empirical investigation. He is making the narrower claim that his examination of the workings of language is not a scientific investigation intended to turn up *new*, perhaps exciting, facts about language; he claims to be dealing only with commonplace facts—matters open to anyone.

Why, then, is Wittgenstein concerned with the workings of language *at all*? How does the activity of describing the commonplace workings of language gain significance? His explicit answer is that such descriptions get their "light, that is to say [their] purpose, from . . . philosophical problems" (*PI* 109). I think this claim is of *first importance* for understanding Wittgenstein's concern with language.

This remark distances Wittgenstein from an approach that has played a dominant role in philosophy for more than a century. Many philosophers have been attracted to the idea that producing a *theory* of language—or, more specifically, developing a theory of *meaning*—is the first task for philosophy. With such a theory in hand, one can then turn to the problems of philosophy, possessing the tools needed for their proper solution or, perhaps, their dissolution. On this approach, theory of meaning comes first; the treatment of other philosophical problems comes later. The logical positivists' attempt to formulate an empiricist criterion of cognitive meaning is one example of this approach. There have been many others. In contrast, I want to suggest that Wittgenstein's concern with philosophy is antecedent to and controls his reflections on language. In the absence of these antecedent philosophical perplexities, I do not think that Wittgenstein would have any philosophical interest (as opposed, say, to a literary interest or philological interest) in language at all. If this is correct, then it is at least misleading to refer to Wittgenstein as a philosopher of language.

Here a comparison between Wittgenstein's later philosophy and J. L. Austin's so-called ordinary language philosophy may be illuminating. Looking at their writings, one is immediately struck, not only by their differences in style, but also by the profound differences in temperament that these stylistic differences reflect. All the same, many of their most basic commitments are similar. Contrary to a long tradition of attributing our philosophical problems to the inaccuracy, vagueness, and crudeness of our everyday language, both Wittgenstein and Austin treat our ordinary or common use of language with respect. Austin and Wittgenstein agree in thinking that it is the philosopher's misunderstanding and misuse

(even abuse) of ordinary language—not ordinary language itself—that generates philosophical confusion.

Austin also shares Wittgenstein's appreciation of the rich diversity (the motley) of the uses of language. Both target the implicit tendency of philosophers to take descriptive language as paradigmatic for understanding the nature of language. In his classic paper "Other Minds," Austin speaks of those who commit the *descriptive fallacy*, as he labels it (Austin 1979, p. 103), which parallels Wittgenstein's attack on what he calls "a particular picture of human language," namely "individual words in language name objects—sentences are combinations of such names" (*PI* 1).

There are, however, important differences between Wittgenstein's and Austin's approaches. Austin thought of himself as a participant in a project involving the "joint labors of philosophers, grammarians, and numerous other students of language" that would yield, sometime in the twenty-first century, "a true and comprehensive science of language" (Austin 1979, p. 232). Wittgenstein would, I think, have little interest in the development of such "a true and comprehensive science of language," even if, contrary to what he believed, it could be successfully pulled off. Such a theory would be an empirical theory, and, as Wittgenstein explicitly states, the philosophical questions that concern him cannot be resolved by an appeal to empirical theories. I will assume that, when he says this, he means it.

The difference between Wittgenstein's and Austin's treatments of philosophical problems comes out in another way. Austin saw our ordinary language as a rich system of finely tuned fields of subtle contrasts. Problems arise, he thought, when philosophers come clomping in with heavy-footed categories that distort these fields,

yielding nonsense in the guise of sense. Philosophical problems are solved or dissolved by a meticulous "teasing out" of fine distinctions. Wittgenstein, in contrast, has little interest in drawing fine distinctions of the kind for which Austin was famous.

> *PI* 106. Here it is difficult as it were to keep our heads up,—to see that we must stick to the subjects of our every-day thinking, and not go astray and imagine that we have to describe extreme subtleties, which in turn we are after all quite unable to describe with the means at our disposal. We feel as if we had to repair a torn spider's web with our fingers.

The task of the philosopher, he tells us, is not to talk "about shades of meaning [where the only thing] in question [is] to find words to hit on the correct nuance" (*PI* 254).[1]

[1] Wittgenstein adds an important qualification to this claim that the philosopher need not engage in subtleties.

> *PI* 254. [Nuances are] in question in philosophy only where we have to give a psychologically exact account of the temptation to use a particular kind of expression. What we 'are tempted to say' in such a case is, of course, not philosophy; but it is its raw material. Thus, for example, what a mathematician is inclined to say about the objectivity and reality of mathematical facts, is not a philosophy of mathematics, but something for philosophical treatment.

Sophistication and subtlety are not needed for recognizing plain facts of linguistic usage. Subtlety and a sense of nuance are, however, needed to unravel the tangled webs philosophers weave in creating complex cognitive illusions. To use a crude analogy, it can be easy to identify a paranoid's basic false belief: He thinks he is being followed by CIA agents. Understanding the mental structures that robustly support this belief may be a subtle, complex problem indeed.

Rather than dwelling on fine distinctions—for example, noting the difference between doing something accidentally and doing it unwittingly—Wittgenstein is usually concerned with broader categories of language: the status of mathematical expressions, of first-person ascriptions of feeling or propositional attitudes, of proper names, general terms, and so on. I do not think Wittgenstein holds that serious philosophical problems arise from confounding terms that lie within a system of contrasting terms. The errors that concern Wittgenstein are not fine-grained errors; they are errors that arise from viewing a whole domain of discourse—a whole genre—in an improper or misleading way, for example, viewing first-person ascriptions of pain on the model of ascriptions of colors to objects, thinking that the only difference is one of subject matter. A parallel mistake occurs with respect to mathematical propositions.

Wittgenstein uses various strategies in dealing with such broad-scale confusions. One is to introduce simple language games of the kind found at the beginning of *Philosophical Investigations*, where generic differences in the uses of language are made transparent.

> *PI* 5. It disperses the fog to study the phenomena of language in primitive kinds of application in which one can command a clear view of the aim and functioning of the words.

He also attempts to exhibit generic differences in uses of language by calling attention to the different ways in which people are trained to employ them. For example, learning how to recite an ordered series of sounds (numerals-to-be) is a standard starting point for learning arithmetic, but nothing similar is used to teach a child

how to ascribe colors to objects. If assertions are under discussion, then various kinds of assertion can be distinguished by examining their modes of verification. For Wittgenstein, all the things he is pointing to are commonplace—things no one doubts, but that we often ignore or consider trite—when engaged in the peculiarly detached business of doing philosophy.

> *PI* 127. The work of the philosopher consists in assembling reminders for a particular purpose.

Another aspect of Wittgenstein's approach, one that seems particularly obnoxious to many philosophers, is what I will call his *defactoism*. I will explain this notion in the next chapter, but the sense of what I have in mind is captured in passages of the following kind:

> *PI* 217. If I have exhausted the justifications I have reached bedrock, and my spade is turned. Then I am inclined to say: "This is simply what I do."

> *OC* 344. My *life* consists in my being content to accept many things.

Passages of this kind are sometimes taken as signs of Wittgenstein's conservatism, pessimism, faintheartedness, or unwillingness to confront serious philosophical issues. Criticisms of this kind treat Wittgenstein's defactoism as an expression of defeat, whereas, instead, defactoism presents a fundamental challenge to the legitimacy of the philosophical enterprise as it is commonly pursued. Defactoism is a central feature of Wittgenstein's later philosophy. Treating it as a basic and unifying theme of his later philosophy is central to this textual study. Chapter 1 is dedicated to this topic.

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I

Rule-Following and the Conceivability of a Private Language

Chapter 1 lays out the central theme of this work: Wittgenstein's rejection, in his later philosophy, of an *interpretational* account of rule-following, replacing it with what I have labeled a *defactoist* account. This theme shapes Wittgenstein's later thought, going back at least to the *Philosophical Grammar*, appearing over and over again in a wide variety of philosophical contexts. This aspect of Wittgenstein's later philosophy has not been wholly ignored—indeed, as noted in the introduction, it has often been decried. Chapter 1 gives it the prominence I think it deserves.

Chapter 2 offers a brisk application of this discussion of rule-following to a question that Wittgenstein raises in *PI* 243:

> Could we imagine a language in which a person could write down . . . individual words [that] refer to what can only be known to the person speaking; to his immediate private sensations. So another person cannot understand the language.

Presented with this question, there is a strong inclination to answer, "Yes, such a language is easy enough to imagine." Countering this, Wittgenstein attempts to show that the rules of this "private language" are mere shadow rules, empty of determinate content.

Chapter One

On Following a Rule

I want to regard man here as an animal; as a primitive being to which one grants instinct but not ratiocination. (*OC* 475)

The Paradox of Interpretation

Readers of *Philosophical Investigations* are familiar with the story of the child being taught to produce the series of even numbers starting with 2. She starts out well enough, writing down 2, 4, 6, 8. However, when asked to pick up the series at 1000, she writes down 1000, 1004, 1008, 1012 (*PI* 183).[1] Told that she is no longer following the instructions we gave her—no longer doing the same thing—she replies that she is, perhaps saying, "Look, see for yourself!" The rub is this: Whatever she writes down, there will be some interpretation of the instructions we gave her—indeed, endlessly many interpretations—such that she has acted in conformity with the rule, and endlessly many interpretations such that she has not. Hence we arrive at what Wittgenstein calls a paradox:

[1] For that matter, she can write down "any damn thing at all," as Wittgenstein at one place puts it. See *LFM*, p. 145.

> *PI* 201. This was our paradox: no course of action could be determined by a rule, because every course of action can be made out to accord with the rule. The answer was: if everything can be made out to accord with the rule, then it can also be made out to conflict with it. And so there would be neither accord nor conflict here.

Saul Kripke opens his second chapter of *Wittgenstein on Rules and Private Language* citing this passage. Then, after a detailed discussion intended to show the force of the paradox, he begins his third chapter with an account of what he thinks Wittgenstein (not the Janus-figure Kripkenstein) is trying to establish.

> The skeptical argument, then, remains unanswered. There can be no such thing as meaning anything by any word. Each new application we make is a leap in the dark; any present intention could be interpreted so as to accord with anything we may choose to do. So there can be neither accord, nor conflict. This is what Wittgenstein said in *PI* 201. (Kripke 1982, p. 55)

Contrary to Kripke's claim, in *PI* 201 Wittgenstein does not say that his paradox shows that there can be no such thing as meaning anything by any word. He does not say this anywhere else. This is what he says instead:

> *PI* 201. It can be seen that there is a misunderstanding here from the mere fact that in the course of our argument we give one interpretation after another; as if each one contented us at least for a moment, until we thought of yet another standing behind it. What this shows is that there is a way of grasping a rule which is *not* an *interpretation*, but which is exhibited in what we

> call "obeying the rule" and "going against it" in actual cases.

Here, Wittgenstein (rather uncharacteristically) is perfectly straightforward in explaining the point he is trying to make. What the paradox shows, he says, "is that there is a way of grasping a rule that is not an interpretation."[2] Surprisingly—actually, incredibly—Kripke never cites this passage in *Wittgenstein on Rules and Private Language* and thus misses what I take to be the central moral of Wittgenstein's paradox: Rule-following cannot be made determinate—or, by extension, meanings cannot be fixed—through interpretation alone.[3]

[2] Gordon Baker and P.M.S. Hacker make this point, saying, "What the absurd paradox that rules cannot guide one *shows* is that how one understands a rule need not be an interpretation, but is manifest *in acting*, in what we call 'following the rule'" (Baker and Hacker 1984, pp. 13–14). I think Wittgenstein is actually saying something stronger—namely, that to avoid the paradox of interpretation, there *has to be* a way of following a rule that is not a matter of interpretation. (This is said explicitly in *PI* 198.)

[3] For Wittgenstein, meaning and rule-following are related in this way: To have a grasp of the meaning of some expression is to have a command of the rules that govern its application. In *Wittgenstein's Lectures: Cambridge, 1932–1935*, Wittgenstein states this connection between meaning and rules in these words:

> The meaning of a word is to be defined by the rules for its use. (*WLC*, p. 3)

Also:

> Two words have the same meaning if they have the same rules for their use. (*WLC*, p. 3)

Wittgenstein is not saying that a person who commands such rules must be able to articulate them—she often cannot. Later he came to see (and stress) that rules themselves can be more or less rigid depending upon the demands of the context in which they are employed. At various places, Wittgenstein warns his reader of the dangers of

Wittgenstein has no brief against rule-following, and no brief against meaning either. He does not think that either rule-following or meaning is inherently paradoxical.[4] His target is a certain account of rule-following (or account of meaning) that, he shows, leads to a paradox. We might call it the *interpretational* account. To fix this firmly in mind, from now on I will talk about Wittgenstein's *paradox of interpretation*. The paradox is this: If we hold that following a rule always involves acting in conformity with an interpretation, then whatever we do will count as both following the rule and not following the rule. Can't this matter be resolved by declaring what interpretation we are acting under? This will not help, for it simply reinstates the paradox of interpretation: What-

idealizing the notion of rules (for example, *PI* 81), but he never, I believe, gives up his commitment to the view that the meaning of an expression is to be explained in terms of the rules governing its use.

[4] In the *Blue Book*, Wittgenstein makes the following remark about the meaning of "meaning":

> "Meaning" is one of the words of which one may say that they have odd jobs in our language. It is these words which cause most philosophical troubles. Imagine some institution: most of its members have certain regular functions, functions which can easily be described, say, in the statutes of the institution. There are, on the other hand, some members who are employed for odd jobs, which nevertheless may be extremely important.—What causes most trouble in philosophy is that we are tempted to describe the use of important 'odd-job' words as though they were words with regular functions. (*B & B*, p. 43)

In contrast to Wittgenstein, Kripke seems to treat "meaning" as a word with a regular referential function and because of this misses, or chooses to ignore, the explicitly stated point of Wittgenstein's treatment of the paradox of rule-following.

ever we say about our intended interpretation will also admit of various interpretations. No interpretation can stop this regress; none has a built-in immunity to further interpretation. There are, we might say, no self-interpreting interpretations.

Just as Wittgenstein is not a skeptic concerning meaning, he has no axe to grind concerning interpretations either—provided, that is, that they are understood as a special kind of activity that takes place *within* language and not as something that lies at its foundation. I take it this is what Wittgenstein has in mind when he says, "We ought to restrict the term 'interpretation' to the substitution of one expression of the rule for another" (*PI* 201). If someone does not act appropriately when instructions are expressed one way, it may help to express them differently. Wittgenstein is not opposed to interpretations understood this way. What he does oppose is the claim (or assumption, or inclination to think) that every meaningful application of a term involves an act of interpretation. Taking him at his word encourages us to see what else Wittgenstein says about the role of interpretation in fixing meaning. What we discover is that the paradox of interpretation is not narrowly tied to Wittgenstein's concerns with private rule-following and the possibility of a private language. The paradox of interpretation is, instead, a recurrent and central feature throughout Wittgenstein's later philosophy. A case can be made for saying that it was part of the original complex of commitments that gave Wittgenstein's later philosophy its characteristic physiognomy. Such a claim is worth showing.

As far as I have been able to discover, the paradox of interpretation makes its first appearance early in *Philo-*

sophical Grammar, a compilation of notes written by Wittgenstein during the years 1932–34.[5] Wittgenstein imagines someone being ordered to square the integers, beginning with 1.

> It seems to us as if by understanding the order we add something to it, something that fills the gap between command and execution. So that if someone said "You understand it, don't you, so it is not incomplete" we could reply "Yes, I understand it, but only because I add something to it, namely the interpretation."—But what makes you give just *this* interpretation? Is it the order? In that case it was already unambiguous, since it demanded this interpretation. Or did you attach the interpretation arbitrarily? In that case what you understood was not the command, but only what you made of it.
>
> (While thinking philosophically we see problems in places where there are none. It is for philosophy to show that there are no problems.)
>
> But an interpretation is something that is given in signs. It is *this* interpretation as opposed to a different one (running differently). So if one were to say "Any sentence still stands in need of an interpretation" that would mean: no sentence can be understood without a rider.
>
> Of course sometimes I do interpret signs, give signs an interpretation; but that does not happen every time I understand a sign. (If someone asks me "What time is it?" there is no inner process of laborious interpretation; I simply react to what I see and hear. If some-

[5] In all likelihood, there are earlier occurrences of this paradox, or at least anticipations of it, that I have not found.

> one whips out a knife at me, I do not say "I interpret that as a threat"). (*PG* I 9)

Though Wittgenstein does not use the term "paradox," the paradox of interpretation is already in place in this work from the early 1930s. The parenthetical remark in the center of this passage indicates that Wittgenstein's therapeutic program of dissolving philosophical problems is also in place.

The closing sentences of the passage from *Philosophical Grammar* anticipate what I now call Wittgenstein's defactoist response to the paradox—our next topic.

A Defactoist Account of Rule-Following

If Wittgenstein rejects the interpretational account of rule-following, what kind of account of rule-following does he put in its place? More specifically, how does he avoid the paradox that confronts the interpretational account? In *Wittgenstein*, I assumed that Wittgenstein was confronted with a genuine paradox and held, alluding to Hume's *Enquiry Concerning Human Understanding*, that he offers only a *skeptical* solution to his paradox.

Hume proceeded in the following way: After arguing in section 4 of the *Enquiry* that it is not possible to provide a rational grounding for inductive inferences, he turns to the task of describing how, despite the lack of a rational grounding, human beings do, after all, form beliefs on the basis of experience. This descriptive activity does not answer the skeptical challenge raised in section 4. Hume did not intend it to. It is in this sense that I took—and still take—Hume's solution to be a *skeptical* solution. In calling Wittgenstein's response to his para-

dox a skeptical solution, I was clearly suggesting that Wittgenstein also despaired of solving his paradox and had no other choice but to fall back on a descriptive activity.[6]

I now think this comparison is in some ways right and in some ways wrong. For Hume, the skeptical argument concerning induction is unanswerable. There is no way of rationally ruling out the possibility that the course of nature might change, so we will forever make our inductive inferences under an irremovable threat. For Hume, the Humean predicament is a human predicament.[7] In contrast, I now think that Wittgenstein does not hold that the paradox of rule-following is unavoidably thrust upon us as something we will have to learn to live with. To put the matter more strongly, for Wittgenstein there is no "paradox" of rule-following. The thought that it is paradoxical is the product of a misconception, namely, the misconception that rule-following is always grounded in (or implicitly contains) acts of interpretation.[8]

[6] There are no references to Hume in Wittgenstein's published material. In a 1930 conversation with M. O'C. Drury, Wittgenstein said in passing that he had never read Hume (Drury 1981, p. 106). In 1946 Wittgenstein remarked to Karl Britton that "he could not sit down and read Hume—he knew far too much about the subject of Hume's writings to find this anything but torture" (Britton 1967, p. 61).

[7] Here, of course, I swipe from Quine's "Epistemology Naturalized" (Quine 1969, p. 72).

[8] In part 2 of *Philosophical Investigations*, Wittgenstein makes a similar remark about the recognition of objects. There is a temptation to think—indeed, many insist on this—that every act of recognizing an object involves a cognitive act of bringing it under some concept. Wittgenstein resists this intellectualizing move:

> One doesn't '*take*' what one knows as the cutlery at a meal *for* cutlery; any more than one ordinarily tries to

Even if Hume sees himself confronted with an unanswerable challenge and Wittgenstein does not, in the tales they tell both of them demote the intellect to a subservient status. Both, to put it quaintly, ground our intellectual capacities in our animal natures. Hume in his *Enquiry Concerning Human Understanding* puts it this way:

> Experimental reasoning itself, which we possess in common with beasts, and on which the whole conduct of life depends, is nothing but a species of instinct or mechanical power, that acts in us unknown to ourselves; and in its chief operations, is not directed by any such relations or comparisons of ideas, as are the proper objects of our intellectual faculties. (*Enquiry*, section 9; Hume 1999, p. 168)

Speaking more specifically about language, Wittgenstein expresses a parallel commitment in these words:

> *OC* 475. I want to regard man here as an animal; as a primitive being to which one grants instinct but not ratiocination. As a creature in a primitive state. Any logic good enough for a primitive means of communication needs no apology from us. Language did not emerge from some kind of ratiocination.

Both Hume and Wittgenstein stress the importance of primitive natural responses shaped through training and other forms of conditioning. This is ground-floor for both. For Hume, going any deeper in exploring the op-

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> move one's mouth as one eats, or aims at moving it. (*PI* II, p. 195)

This, I think, is of a piece with what Wittgenstein is saying about the role of interpretation in rule-following.

erations of the human understanding is beyond our intellectual capacities. For Wittgenstein, it is an illusion to suppose that there has to be something deeper, even if hidden, that could provide such an explanation. This is an important difference—one that I did not formerly appreciate sufficiently. Still, important similarities remain, as we shall see in examining Wittgenstein's detailed remarks on rule-following.

The Communitarian versus the Defactoist Account of Rule-Following

How, precisely, does Wittgenstein give an account of rule-following that avoids the difficulties found in the interpretational account? An answer is sketched in *PI* 198—a section of central importance for understanding Wittgenstein's account of rule-following, and one, I think, that has often been misunderstood. It begins with a statement of the paradox of interpretation:

> *PI* 198. "But how can a rule show me what I have to do at *this* point? Whatever I do is, on some interpretation, in accord with the rule."—That is not what we ought to say, but rather: any interpretation still hangs in the air along with what it interprets, and cannot give it any support. Interpretations by themselves do not determine meaning.

Then Wittgenstein has an interlocutor intervene, asking a perfectly natural question:

> "Then can whatever I do be brought into accord with the rule?"

Speaking in his own voice, Wittgenstein responds:

> Let me ask this: what has the expression of a rule—say a sign-post—got to do with my actions? What sort of connexion is there here? Well, perhaps this one: I have been trained to react to this sign in a particular way, and now I do so react to it.

Wittgenstein speaks for the interlocutor and raises a further objection:

> But that is only to give a causal connexion; to tell how it has come about that we now go by the sign-post; not what this going-by-the-sign really consists in.

Wittgenstein responds:

> On the contrary; I have further indicated that a person goes by a sign-post only in so far as there exists a regular use of sign-posts, a custom.[9]

For reasons I do not understand, the closing sentence, with its reference to custom, seems to cloud the minds of many commentators with respect to the references to training that precede it. It also induces a subsequent blindness to the frequent references to training that will follow. Although there is overwhelming textual support—passage after passage—for assigning a central role to natural responses and training in Wittgenstein's account of rule-following, many commentators assign rela-

[9] Internal dialogues of this kind occur throughout Wittgenstein's later writings. They serve a variety of purposes. With few exceptions, the interlocutor is not a stooge but, instead, represents someone who raises objections or makes suggestions that arise *naturally* when one is a captive of a philosophical misunderstanding or not fully free of it. Often, though not always, the dialogue can be seen as an exchange between the early Wittgenstein and the late Wittgenstein. We will examine an extended example of such an exchange later in this chapter. For a detailed examination of these internal dialogues, see Stern 2004.

tively little importance to them and have instead taken a quite different approach, maintaining that Wittgenstein attempts to deal with this paradox by invoking some form of *communitarianism*.

The communitarian response to the paradox can be implemented in various ways. Broadly speaking, it goes something like this. The community provides what an isolated speaker cannot provide: an independent standard for determining whether a rule has been followed correctly or not. This can be spelled out in various ways. We might insist that an individual interpret the rule as members of the community interpret it, or at least insist that the individual's action conform to the rule as the community interprets it. It is, however, hard to see how such a maneuver will get us out of our difficulties, for the paradox of interpretation breaks out anew, now at the community level.[10] Whatever the members of the community do, or say they are doing, under some interpretations of their rules their actions will conform to them, and under others they will not. Wittgenstein's claim that "there is a way of grasping a rule which is *not* an *interpretation*" is not restricted in its scope. It applies to individuals and communities alike. Applied to communities of language users, the moral to the paradox of interpretation is that there must be a way for the community to grasp a rule that is not an interpretation—communal or otherwise. As we shall see, communitarianism does have

[10] The worry that the paradox can break out anew at the community level has been expressed, on various grounds, by a number of people. I expressed it in *Wittgenstein* (Fogelin 1976b and 1987). Many have raised the same objection since; for example, Hilary Putnam, "On Wittgenstein's Philosophy of Mathematics" (Putnam 1996). I would be surprised if others had not raised it before I did.

an important role to play in Wittgenstein's understanding of rule-following, but it does not, by itself, provide the remedy for the paradox of interpretation.

The point I am trying to make concerning communitarianism finds explicit statement in the following remarkable passage from the transcript of Wittgenstein's 1939 Cambridge *Lectures on the Foundations of Mathematics*:

> [Consider] the question of how to continue the series of cardinal numbers. Is there a criterion for the continuation—for a right and a wrong way—except that we do in fact continue them in that way, apart from a few cranks who can be neglected?
>
> We do indeed give a general rule for continuing the series; but this general rule might be reinterpreted by a second rule, and this second rule by a third rule, and so on. . . .

Then this:

> [It] has often been put in the form of an assertion that the truths of logic are determined by a consensus of opinions. Is this what I am saying? No. There is no *opinion* at all; it is not a question of *opinion*. They are determined by a consensus of *action*: a consensus of doing the same thing, reacting in the same way. There is a consensus but it is not a consensus of opinion. We all act the same way, walk the same way, count the same way. (*LFM*, pp. 183–84)

The second paragraph contains a thumbnail version of Wittgenstein's paradox of interpretation. The surrounding material contains what I call Wittgenstein's defactoist response to it. Wittgenstein is not saying that an individ-

ual's interpretation of a rule is correct to the extent that it squares with the community's interpretation of it. In rule-following, we join a consensus in *action*—a consensus grounded in the kind of training that we, as humans, *can* successfully undergo and the kind of training that we actually *do* undergo in the community in which we are reared. The consensus is grounded, as Wittgenstein puts it, in facts concerning our *natural history*. I have in mind passages of the following kind:

> *PI* 25. It is sometimes said that animals do not talk because they lack the mental capacity. And this means: "they do not think, and that is why they do not talk." But—they simply do not talk. Or to put it better: they do not use language—if we except the most primitive forms of language.—Commanding, questioning, recounting, chatting, are as much a part of our natural history as walking, eating, drinking, playing.[11]

> *PI* 415. What we are supplying are really remarks on the natural history of human beings; we are not contributing curiosities however, but observations which no one has doubted, but which have escaped remark only because they are always before our eyes.

The central role that Wittgenstein assigns to these facts of our natural history constitutes what I am calling Wittgenstein's *defactoism*. "Defactoism" is, I know, an

[11] This passage is strikingly similar to one found in Hume's *Treatise of Human Nature*:

> Nature, by an absolute and uncontroulable necessity, has determin'd us to judge as well as to breathe and feel. (Hume 2000, p. 123)

ugly word.[12] It is also, as far as I know, a new word. I have adopted it for two reasons. First, in its bluntness it is descriptively apt in characterizing a central feature of Wittgenstein's mode of philosophizing. Second, being new, it does not carry the freight of other labels often employed in discussions of Wittgenstein's philosophy, such as antifoundationalism, constructivism, finitism, behaviorism, and so on. I will not attempt to offer a definition of defactoism but, instead, explore the various interrelated themes that make it up.

For Wittgenstein, the root error of much philosophizing is to press on in search of reasons where are none to be found.

> *Z* 314. Here we come up against a remarkable and characteristic phenomenon in philosophical investigation: the difficulty—I might say—is not that of finding the solution but rather that of recognizing as the solution something that looks as if it were only a preliminary to it. "We have already said everything.—Not anything that follows from this, no, *this* itself is the solution!"
>
> This is connected, I believe, with our wrongly expecting an explanation, whereas the solution of the difficulty is a description, if we give it the right place in our considerations. If we dwell upon it, and do not try to get beyond it.
>
> The difficulty here is: to stop.

[12] The Latin phrase "*quid facti*" sounds more respectable and calls to mind Kant's contrasting phrase "*quid juris*." This terminology might be used to explain the fundamental difference between Wittgenstein's early and later philosophy. I'll resist this temptation.

> *PI* 654. Our mistake is to look for an explanation where we ought to look at what happens as a 'proto-phenomenon'. That is, where we ought to have said: *this language-game is played*.

The Respective Roles of Training and Community in Rule-Following

Let me venture an example illustrating what I take to be the relationship between Wittgenstein's communitarianism and his defactoism. For whatever reason, Burris has trained his son Michael to hum (not sing) the melody of "It's a Grand Old Flag" whenever he sees an American flag—and only then. The training began when Michael was very young and the reaction is now "hardwired" into him. Now suppose his father hears Michael humming "It's a Grand Old Flag." From this, his father—or anyone familiar with the conditioning Michael received—could reasonably infer that Michael has seen an American flag. (Finding that he is humming the melody, Michael himself might infer that he must have caught a glimpse of Old Glory, perhaps without attending to it.) Do we want to say that Michael's humming means "There's an American flag"? No! Not unless his humming has been assigned that role in a language. Do we want to say that his humming is at least an instance of following a rule? Even that would be misleading, for rules, as commonly understood, are *of* something. There are rules of chess, of baseball, of the road, of etiquette, and so on. Rules (*Regeln*), for Wittgenstein, are regulations or akin to regulations: They govern people involved in practices, activities with a purpose or a point. Wittgenstein is concerned with the community-embedded training that our actual languages

(and other institutions) embody. Michael's humming does not have this status.[13]

Wittgenstein adopts what we might call a *rich* notion of rule-following, and with it a rich notion of what it is to be a language. An important aspect of this rich notion of rule-following is that rules governing a practice hang together and interrelate in ways that reflect the purpose or purposes of the practice. Because of this, some rules are more essential or central than others.

> *PI* 564. [I] am inclined to distinguish between the essential and the inessential in a game too. The game, one would like to say, has not only rules but also a *point*.

> *PI* 567. But [a] game is supposed to be defined by [its] rules! So, if a rule of the game prescribes that the kings are to be used for drawing lots before a game of chess, then that is an essential part of the game. What objection might one make to this? That one does not see the point of this prescription. . . .
>
> ((Meaning is a physiognomy.))

It is, I think, important to keep Wittgenstein's rich conception of rule-following in mind when dealing with "communitarian" passages of the following kind:

[13] After making up this example, I found the following parallel example in *Philosophical Investigations*:

> If you trained someone to emit a particular sound at the sight of something red, another at the sight of something yellow, and so on for other colours, still he would not yet be describing objects by their colours. Though he might be a help to us in giving a description. A description is a representation of a distribution in a space (in that of time, for instance). (*PI* II, p.187)

My guess is that my example has its source in a dim recollection of this passage.

> *PI* 199. Is what we call "obeying a rule" something that it would be possible for only *one* man to do, and to do only *once* in his life?—This is of course a note on the grammar of the expression "to obey a rule".[14]
>
> It is not possible that there should have been only one occasion on which someone obeyed a rule. It is not possible that there should have been only one occasion on which a report was made, an order given or understood; and so on.—To obey a rule, to make a report, to give an order, to play a game of chess, are *customs* (uses, institutions).

The same rich notion of rule-following occurs in this passage from *Remarks on the Foundations of Mathematics:*

> *RFM* VI 21. The application of the concept 'following a rule' presupposes a custom. Hence it would be nonsense to say: just once in the history of the world someone followed a rule (or a signpost; played a game, uttered a sentence, or understood one; and so on).

For something to be a rule, it must have a point, and to have a point it must be embedded in practices (customs, institutions). This is how Wittgenstein understands the notion of a rule. It is also, I believe, the common way of understanding what a rule is.

As the passage just cited continues, it shifts seamlessly into a defactoist mode:

> Here there is nothing more difficult than to avoid pleonasms and only to say what really describes something.

[14] In the next chapter I will look closely at what Wittgenstein has in mind in calling his remark "a note on the grammar of the expression 'to obey a rule.'"

> For here there is an overwhelming temptation to say something more, when everything has already been described.
>
> It is of the greatest importance that a dispute hardly ever arises between people about whether the colour of this object is the same as the colour of that, the length of this rod the same as the length of that, etc. This peaceful agreement is the characteristic surrounding of the use of the word "same."
>
> And one must say something analogous about proceeding according to a rule.
>
> No dispute breaks out over the question whether a proceeding was according to the rule or not. It doesn't come to blows, for example.
>
> This belongs to the framework, out of which our language works (for example, gives a description).

Wittgenstein makes a similar remark in *PI* 242:

> *PI* 242. If language is to be a means of communication there must be agreement not only in definitions but also (queer as this may sound) in judgments. This seems to abolish logic, but does not do so.—It is one thing to describe methods of measurement, and another to obtain and state results of measurement. But what we call "measuring" is partly determined by a certain constancy in results of measurement.

People are trained in the procedures for making measurements. Mistakes are, of course, possible—hence the carpenter's precept, "Measure twice, cut once." Disagreements can occur but, in fact, rarely do. Why is this? What is the ground of this common agreement? Wittgenstein's response, to repeat it, is to resist what he calls the over-

whelming temptation "to say something more, when everything has already been described." [15]

There are cases where an amiable consensus does not arise. Quarrels *do* sometimes break out concerning which paint sample best matches a painted surface. But suppose that such disagreements happened constantly for all color ascriptions: What would determine who was right and who was wrong? Well, they do not happen constantly! That's the fact of the matter.

To my mind, Wittgenstein's repeated appeals to training have been underappreciated. To counter this, here are further examples of such appeals drawn from various sources. The *Brown Book* opens with an examination of the slab-beam language game that would appear later in section 2 of *Philosophical Investigations*. It contains this remark concerning the sort of training—at least for the primitive language game—he has in mind:

> I am using the word "trained" in a way strictly analogous to that in which we talk of an animal being trained to do certain things. It is done by means of example, reward, punishment, and suchlike. (*B & B*, p. 77)

Wittgenstein also indicates that the kind of training that animals, including human animals, can undergo will depend on the sorts of creatures they are. Again the *Brown Book*:

> Imagine . . . that you tried to teach a cat to retrieve. As the cat will not respond to your encouragement, most of the acts of encouragement which you performed when you trained the dog are here out of the question. (*B & B*, p. 90)

[15] In *Remarks on the Philosophy of Psychology*, Wittgenstein speaks of the temptation "to talk more than makes sense" (*RPP* vol. 2, 402).

A similar passage occurs in *Wittgenstein's Lectures: Cambridge, 1932–1935*:

> A calf or a cat cannot be taught [to look after sheep]; I could go through all the motions with these animals and would not get an appropriate reaction. Training can be described as consisting of two steps, (1) the trainer's doing certain things, (2) the occurrence of certain reactions on the part of the subject, with the possibility of improvement. Teaching a language always depends on a training which presupposes that the subject reacts. If the subject does not react in a given case, that is, does not understand, reference to understanding will then not appear in the description of the training. But nothing is omitted from the description by omitting reference to understanding. (*WLC*, p. 102)

This, then, is one important aspect of what I am calling Wittgenstein's defactoism: The kind of training a creature is capable of undergoing will depend on the repertoire of natural or instinctive responses the creature possesses. Of course, the particular skills that the creature will master depend on the actual training it receives. Most dogs are not trained to herd sheep. Some are trained to jump through flaming hoops. Some, if their behavior is any indication, receive no significant training at all. The situation is the same for human creatures. They too have a repertoire of natural responses that can be shaped through training. The particular rules that they are trained to follow are in large measure a function of the society they inhabit. It is in this way that communitarianism plays an important role in Wittgenstein's account of rule-following.

A second, and central, aspect of Wittgenstein's defactoism, as I am using the notion, is his rejection of the idea

that training is merely an *external* device intended to induce in the trainee a grasp of the correctness, the legitimacy, of what he has been trained to do. This passage comes from the *Lectures on the Foundations of Mathematics:*

> The only criterion [for a student's] multiplying 113 by 44 . . . is his doing it in the way in which all of us, who have been trained in a certain way, would do it. If we find that he cannot be trained to do it the same as us, then we give him up as hopeless and say he is a lunatic. (*LFM*, p. 58)

Here Wittgenstein says that the *only* criterion for the student's multiplying correctly is conformity to public practice. Those who do the training have themselves been trained. The student who cannot conform to their training is dismissed as a hopeless case—a lunatic. There is nothing more to it than that. In particular, the training need not produce a mental intermediary (a third thing) in virtue of which the student, in concert with members of his community, is able to multiply correctly. Or rather, even if training did produce some such mental intermediary, it would not provide a justification of the student's performance. Appeals to it would simply raise the paradox of interpretation anew.

This nothing-more move is a recurrent theme throughout Wittgenstein's later writings. It gets a forceful statement in *Wittgenstein's Lectures: Cambridge, 1932–1935*, where the paradox of interpretation and his defactoist response to it are in full flower. The example again concerns a student being taught to produce a series of numbers, this time by steps of 10.

> Suppose now that he is ordered to add 10, and that the highest number reached in the training is 100.

> Upon being given the order he writes 10, 20, . . . 100, 120, 140, 160, and the teacher objects that he did not carry out the order. But why? The teacher replies that he was meant to do this: 100, 110, . . . 1,000,000, 1,000,010, and when did the teacher mean it? When he trained him. . . . The pupil is given a rule and examples, and the teacher may say that he *means* something, that though not stated is conveyed indirectly by means of these. It would seem that if what is meant could be conveyed, and not merely the clumsy rule and examples, he could be *made* to continue with 110 after 100. *But the teacher also has only the rule and examples. It is a delusion to think that you are producing the meaning in someone's mind by indirect means, through the rule and examples*. (*WLC*, pp. 131–32, emphasis added)

We find passages of the same kind in *Philosophical Investigations*. A defactoist moment in Wittgenstein's description of how we explain the meaning of such abstract words as "regular," "uniform," and "same" illustrates this theme.[16]

> *PI 208*. I shall explain these words to someone who, say, only speaks French by means of the corresponding French words. But if a person has not yet got the *concepts*, I shall teach him to use the words by means of *examples* and by *practice*.—And when I do this I do not communicate less to him than I know myself.

Here the person doing the teaching possesses certain skills and abilities. She can give examples of uniformities and nonuniformities. She can employ the concept in a

[16] These words, "regular," "uniform," and "same" are particularly of interest because of their connection with rule-following.

variety of contexts. In her teaching, she is trying to imbue her student with these same skills and abilities. She is not holding anything back. She is not in possession of a secret key that she is trying to pass on to her student that, when successfully transmitted, will successfully complete the training.

A similar defactoist response to a variant of the paradox of interpretation occurs in Wittgenstein's much-examined discussion of family-resemblance concepts:

> *PI* 71. This is . . . how one might explain to someone what a game is. One gives examples and intends them to be taken in a particular way.—I do not, however, mean by this that he is supposed to see in those examples that common thing which I—for some reason—was unable to express; but that he is now to *employ* those examples in a particular way. Here giving examples is not an *indirect* means of explaining—in default of a better. For any general definition can be misunderstood too. The point is that this is how we play the game. (I mean the language-game with the word "game".)

Of course, it will not be possible for the teacher to convey the *essence* of a game if there is none. That, however, is not the point that Wittgenstein is making in this passage. What he is saying comes to this: The person being taught what a game is may fail to employ the example in a proper way, but the same can happen in cases where there *is* a "common thing" that all items that fall under a concept share.

The moral we are supposed to draw from this passage and from other passages of the same kind that I have cited seems clear: Efforts to find some cognitive (ratio-

nal, intellectual, ideational—pick your poison) intermediary or "third man" that will establish a secure connection between an expression and its application are misconceived. The paradox of interpretation dooms all such efforts.

Before closing this discussion of intermediaries, it will be worth noting that Wittgenstein's rejection of interpretations as mental intermediaries is of a piece with Wittgenstein's *general* distaste for mental intermediaries. A common, though unsustainable, idea is that images connect words to things. At its crudest, this is the notion that we think about external objects by entertaining images of them. Part of Wittgenstein's response to this suggestion is that images, like pictures, admit of various applications, so unless the application is somehow fixed, the appeal to an image yields another version of the paradox of interpretation. What we are hankering after is an image or some other form of representation that fixes its own application. Wittgenstein has his interlocutor express just this demand:

> *PI* 389. "The image must be more like its object than any picture. For, however like I make the picture to what it is supposed to represent, it can always be the picture of something else as well. But it is essential to the image that it is the image of *this* and of nothing else."

Wittgenstein brushes this aside with these words:

> Thus one might come to regard the image as a super-likeness [*Über-Bildnis*].[17]

[17] This mocking of the philosopher's demand for a super-likeness is of a piece with the attitude that Wittgenstein takes toward other su-

He illustrates this point in a marginal note inserted in the region of *PI* 139:

> I see a picture; it represents an old man walking up a steep path leaning on a stick.—How? Might it not have looked just the same if he had been sliding downhill in that position? Perhaps a Martian would describe the picture so. I do not need to explain why *we* do not describe it so.

Why doesn't he have to explain why we do not describe it so? Presumably because the answer is obvious: We automatically, unreflectively, associate the picture with the familiar phenomenon of someone walking up a hill rather than with the rare phenomenon of someone, perfectly in balance, sliding backward down a hill. (Think what it would be like to represent someone as sliding backward down a hill.)

Wittgenstein has another objection to the supposed explanatory force of a word-image-object relationship: It is no less mysterious than the word-object relationship it is intended to explain. This passage comes from the *Brown Book*:

> B learnt to bring a building stone on hearing the word "column!" called out. We could imagine what happened in such a case to be this: In B's mind the word called out brought up an image of a column, say; the training had, as we should say, established this associa-

per-notions: super-rules, *Über-Regeln* (*PG* VI 72); a super-expression, *Über-Ausdruck* (*RFM* I 24 and *PI* 192); a super-strong-connection, *über-starre Verbindung*, that supposedly holds between intentions and actions (*RFM* I 130 and *PI* 197); and the super-order, *Über-Ordnung*, between super-concepts, *Über-Begriffen* (*PI* 97). Such "philosophical superlatives" (*PI* 192) are a persistent target of Wittgenstein's ridicule.

> tion. B takes up that building stone which conforms to his image.—But was this *necessarily* what happened? If the training could bring it about that the idea or image—automatically—arose in B's mind, why shouldn't it bring about B's *actions* without the intervention of an image? This would only come to a slight variation of the associative mechanism. Bear in mind that the image which is brought up by the word is not arrived at by a rational process (but if it is, this only pushes our argument further back), but that this case is strictly comparable with that of a mechanism in which a button is pressed and an indicator plate appears. In fact this sort of mechanism can be used instead of that of association. (*B & B*, p. 89)

What I am calling Wittgenstein's defactoism appears in various guises, and I do not think I can adequately define it except for saying that it involves, in a variety of ways, the rejection of appeals to rational processes where philosophers typically have attempted to find or supply them. The final two sentences of the above passage vividly exemplify what I have in mind.

The Ineffability of Rule-Following

So far my account of Wittgenstein's paradox differs from Kripke's in two ways: First, we present the paradox differently; and second, we describe Wittgenstein's treatment of the paradox differently. But there is a third important difference: Kripke examines the paradox of rule-following, paying relatively little attention to Wittgenstein's reflections on the nature of rules and on the nature of rule-following. Passages of the following kind,

though crucial to Wittgenstein's conception of rule-following, get scant attention from Kripke:

> *RFM* VI 31. To what extent can the function of language be described? If someone is not master of a language, I may bring him to a mastery of it by training. Someone who is master of it, I may remind of the kind of training, or I may describe it; for a particular purpose; thus already using a technique of the language.
>
> To what extent can the function of a rule be described? Someone who is master of none, I can only train. But how can I explain the nature of a rule to myself?
>
> The difficult thing here is not, to dig down to the ground; no, it is to recognize the ground that lies before us as the ground.
>
> For the ground keeps on giving us the illusory image of a greater depth, and when we seek to reach this, we keep on finding ourselves on the old level.
>
> Our disease is one of wanting to explain.

The same point is made in *Zettel*.

> *Z* 318. I cannot describe how (in general) to employ rules, except by *teaching* you, *training* you to employ rules.

Moreover, if we observe our actions from, as it were, the "outside," the sense of rule-following—the sense of normativity—can simply evaporate. Wittgenstein captures this in a pretty image in *Philosophical Grammar*:

> Let us imagine we are sitting in a darkened cinema and entering into the happenings in the film. Now the

> lights are turned on, though the film continues on the screen. But suddenly we see it "from outside" as movements of light and dark patches on a screen. (*PG* I 98)

This reference to "entering into" something as opposed to "seeing it 'from outside'" calls to mind the agent/observer (inner/outer) contrast central to the Kantian tradition. Wittgenstein was presumably familiar with this contrast from his youthful reading of Schopenhauer's philosophy, where it plays a central role. It also seems that he was quite taken with it during his Tractarian period. It is, however, hard to know what to make of his use of this contrast in the context of *Philosophical Grammar*. Wittgenstein puts the expression "from outside" in quotation marks, perhaps flagging it as suspect. In *Philosophical Investigations* he issues an explicit warning against assigning any explanatory value to this contrast:

> *PI* 631. "I am going to take two powders now, and in half-an-hour I shall be sick."—It explains nothing to say that in the first case I am the agent, in the second merely the observer. Or that in the first case I see the causal connexion from inside, in the second from outside. And much else to the same effect.

The spatial imagery of inner vs. outer is, we might say, a picture that naturally comes to mind when we try to articulate the contrast between acting under a rule and acting in a way that just happens to conform to a rule.

Reflections on the ineffability of rule-following are captured in these important passages:

> *PI* 175. Make some arbitrary doodle on a bit of paper.—And now make a copy next to it, let yourself be

> guided by it.—I should like to say: "Sure enough, I was guided here. But as for what was characteristic in what happened—if I say what happened, I no longer find it characteristic."
>
> But now notice this: *while* I am being guided everything is quite simple, I notice nothing *special*; but afterwards, when I ask myself what it was that happened, it seems to have been something indescribable. *Afterwards* no description satisfies me. It's as if I couldn't believe that I merely looked, made such-and-such a face, and drew a line. . . .
>
> *PI* 176. When I look back on the experience I have the feeling that what is essential about it is an 'experience of being influenced', of a connexion—as opposed to any mere simultaneity of phenomena: but at the same time I should not be willing to call any experienced phenomenon the "experience of being influenced". (This contains the germ of the idea that the will is not a *phenomenon*.) I should like to say that I had experienced the '*because*', and yet I do not want to call any phenomenon the "experience of the because".

These passages contain flags routinely used by Wittgenstein to indicate that what is being said cannot be taken straight. Here he puts words in what we might call *warning quotation marks* and, more particularly, he uses the phrase "I should like to say." At a certain point in our philosophical reflections we find ourselves naturally inclined to say that there really is a something—"a because"—guiding our actions. But it also strikes us as an indescribable something. Not only that, its guiding force seems to dissipate when we make it an object of examination.

From the Sublime to the Mundane

There is a further aspect of what I am calling Wittgenstein's defactoism that can be brought out by contrasting the conception of rules found in the *Tractatus* with the standpoint he adopted in his later philosophy. In the *Tractatus*, the rules underlying our language are thought to mirror the eternal, unchanging, necessary structure of the world. The rules that allow us to represent the world have to be correspondingly strict, unalterable, and wholly determinate. Looking back on the *Tractatus*, Wittgenstein describes how one might be led to adopt such a standpoint. It is the result of being taken in and being dominated by a misleading comparison.

> *PI* 81. In philosophy we often *compare* the use of words with games and calculi which have fixed rules, but cannot say that someone who is using language *must* be playing such a game.—But if you say that our languages only *approximate* to such calculi you are standing on the very brink of a misunderstanding. For then it may look as if what we were talking about were an *ideal* language. As if our logic were, so to speak, a logic for a vacuum.— . . . But here the word "ideal" is liable to mislead, for it sounds as if these languages were better, more perfect, than our everyday language; and as if it took the logician to shew people at last what a proper sentence looked like.

Wittgenstein came to abandon his earlier notion of rule-following, because, if we look at the way language is *actually* employed, we see at once that it comes nowhere near meeting the logician's ideal standards, yet, for all

that, it serves a wide range of useful purposes. Ordinary language, for the most part, does not exhibit the rigid rules that logicians demand and, for the most part, stands in no need of them. This rejection of the demand for the rigidity of rules goes hand in hand with the rejection of the doctrine of definiteness of sense. The lines are drawn between his former conceptions of language and his new conception of language in the following passage, which begins a sustained dialogue between Wittgenstein the elder and Wittgenstein the younger.

> *PI* 65. Here we come up against the great question that lies behind all these considerations.—For someone might object against me: "You take the easy way out! You talk about all sorts of language-games, but have nowhere said what the essence of a language-game, and hence of language, is: what is common to all these activities, and what makes them into language or parts of language. So you let yourself off the very part of the investigation that once gave you yourself most headache, the part about the *general form of propositions* and of language."

Here the younger Wittgenstein is scolding the older Wittgenstein for his lack of courage in facing up to questions that previously absorbed his full attention. Wittgenstein the elder responds:

> And this is true.—Instead of producing something common to all that we call language, I am saying that these phenomena have no one thing in common which makes us use the same word for all,—but that they are *related* to one another in many different ways. And it is because of this relationship, or these relationships, that we call them all "language".

The central move in Wittgenstein's response to the charge that he is taking the "easy way out" is to challenge the essentialist presupposition that lies behind it. An examination of our *actual* use of language will show that, in fact, things can be properly subsumed under the same term without there being a defining quality common to them all. Since Wittgenstein has been speaking about language games, he, aptly enough, illustrates this point with respect to games themselves.

> *PI* 66. Don't say: "There *must* be something common, or they would not be called 'games' "—but *look and see* whether there is anything common to all.—For if you look at them you will not see something that is common to *all*, but similarities, relationships, and a whole series of them at that. To repeat: don't think, but look!

In *PI* 67 Wittgenstein introduces the notion of *family resemblance* as a suitable metaphor for systems that are held together by crisscrossing and overlapping similarities with no defining feature running through them all. He illustrates the notion of family resemblance with respect to a variety of topics, including numbers (*PI* 67–68), plants (*PI* 70), leaves (*PI* 73), and goodness (*PI* 77). He then presents a parallel account of how proper names function. Just as the employment of general terms need not depend on the existence of shared essences, we can, and often do, employ a proper name such as "Moses" without giving it a fixed meaning, say, by ascribing an individual essence to Moses himself (*PI* 79).[18]

These anti-essentialist themes are a central feature of

[18] Wittgenstein presents an account of proper names that starts with Russell's idea that proper names should be analyzed in terms of definite descriptions and then goes off on its own by modifying Russell's approach in the direction of indeterminacy.

Wittgenstein's later philosophy and have been given prominence in the scholarly literature (including my own writings). I do not wish to underestimate their importance. The point I wish to stress is that Wittgenstein's anti-essentialism is one aspect of his more general attack on the related demands for rigid rules and definite sense. For example, in holding that numbers, like games, are bound together only as a system of family resemblances, Wittgenstein acknowledges that we could, if we so choose, place sharp boundaries on the concept of a number by defining it, say, as the logical sum of those things that now count as numbers: cardinal numbers, rational numbers, real numbers, and the like.

PI 79. We may say, following Russell: the name "Moses" can be defined by means of various descriptions. For example, as "the man who led the Israelites through the wilderness", "the man who lived at that time and place and was then called 'Moses' ", "the man who as a child was taken out of the Nile by Pharaoh's daughter," and so on. And according as we assume one definition or another the proposition "Moses did not exist" acquires a different sense, and so does every other proposition about Moses.—And if we are told "N did not exist", we do ask: "What do you mean? Do you want to say . . . . . or . . . . . etc.?"

But when I make a statement about Moses,—am I always ready to substitute some *one* of these descriptions for "Moses"? I shall perhaps say: By "Moses" I understand the man who did what the Bible relates of Moses, or at any rate a good deal of it. But how much? Have I decided how much must be proved false for me to give up my proposition as false? Has the name "Moses" got a fixed and unequivocal use for me in all possible cases?—Is it not the case that I have, so to speak, a whole series of props in readiness, and am ready to lean on one if another should be taken from under me and vice versa? . . .

> *PI* 68. I *can* give the concept 'number' rigid limits . . . that is, use the word "number" for a rigidly limited concept, but I can also use it so that the extension of the concept is *not* closed by a frontier. And this is how we do use the word "game". For how is the concept of a game bounded? What still counts as a game and what no longer does? Can you give the boundary? No. You can *draw* one; for none has so far been drawn. (But that never troubled you before when you used the word "game".)

Concepts lacking sharp boundaries leave open what are called *borderline cases*. Sometimes, for example, we are not prepared to say whether something is a game or not. This, however, need not silence us. We can say that it is like a game in some ways but not like one in other ways.

> And this can be expressed like this: I use the name "N" without a *fixed* meaning. (But that detracts as little from its usefulness, as it detracts from that of a table that it stands on four legs instead of three and so sometimes wobbles.)

But is it true that a user of the name "Moses" has "a whole series of props in readiness, and [is] ready to lean on one if another should be taken from under [him]"? That, contrary to Wittgenstein's suggestion, seems factually implausible. It seems more reasonable, as Wittgenstein also suggests, to say that those who believe that Moses actually existed take it that the biblical account of his life is sufficiently reliable to buttress such a belief. They can believe this without being able to remember many of the details of the narrative. Furthermore, this belief can be sustained even if some of the episodes in the story are shown to be fabrications. (This, of course, is not an issue for those who believe in the inerrancy of the Bible.) As more fabrications are exposed, the credibility of the narrative as a whole decreases. Eventually Moses might find himself in the company of Paul Bunyan and the Tooth Fairy. Need there be a sharp dividing line that would mark this demotion? Wittgenstein would find this factually implausible.

There are, however, even more radical forms of indeterminacy where we would be at a loss to know what to say at all. They do not involve borderline cases but what we might call *bizarre* cases.

> *PI* 80. I say "There is a chair". What if I go up to it, meaning to fetch it, and it suddenly disappears from sight?—"So it wasn't a chair, but some kind of illusion".—But in a few moments we see it again and are able to touch it and so on.—"So the chair was there after all and its disappearance was some kind of illusion."—But suppose that after a time it disappears again—or seems to disappear. What are we to say now? Have you rules ready for such cases—rules saying whether one may use the word "chair" to include this kind of thing? But do we miss them when we use the word "chair"; and are we to say that we do not really attach any meaning to this word, because we are not equipped with rules for every possible application of it?

Now there is nothing original in noting that the rules that govern the ways in which we actually employ language often fall short—often far short—of the logician's ideals. Historically, it has been customary for philosophers to deplore the shortcomings of everyday language and then attempt to put something better in its place. In the *Tractatus*, Wittgenstein took quite a different tack. He held that "all the propositions of our everyday language, just as they stand, are in perfect logical order" (*TLP* 5.5563) in virtue of an ideal, intermediate structure that properly connects the underlying structure of everyday language with the underlying structure of the world. In *PI 89*, Wittgenstein confronts this Tractarian picture by posing the question, "In what sense is logic something

sublime?" What follows is another dialogue between Wittgenstein and his earlier, Tractarian, self. I will pick it up at *PI* 94.

> *PI* 94. "A proposition is a queer thing!" Here we have in germ the subliming of our whole account of logic. The tendency to assume a pure intermediary between the propositional *signs* and the facts. Or even to try to purify, to sublime, the signs themselves.—For our forms of expression prevent us in all sorts of ways from seeing that nothing out of the ordinary is involved, by sending us in pursuit of chimeras.

Later, using "we are" as a stand-in for "I and others were," he continues:

> *PI* 97. We are under the illusion that what is peculiar, profound, essential, in our investigation, resides in its trying to grasp the incomparable essence of language. That is, the order existing between the concepts of proposition, word, proof, truth, experience, and so on. This order is a *super*-order between—so to speak—*super*-concepts.

Then, speaking in his own voice, Wittgenstein rejects this picture:

> Whereas, of course, if the words "language", "experience", "world", have a use, it must be as humble a one as that of the words "table", "lamp", "door".

In *PI* 99 Wittgenstein continues this dialogue with his former self, again speaking from the Tractarian standpoint:

> *PI* 99. The sense of a sentence—one would like to say—may, of course, leave this or that open, but the

> sentence must nevertheless have *a* definite sense. An indefinite sense—that would really not be a sense *at all*. . . .
>
> *PI* 102. The strict and clear rules of the logical structure of propositions appear to us as something in the background—hidden in the medium of the understanding. I already see them (even though through a medium): for I understand the propositional sign, I use it to say something.
>
> *PI* 103. The ideal, as we think of it, is unshakable. You can never get outside it; you must always turn back. There is no outside; outside you cannot breathe.—Where does this idea come from? It is like a pair of glasses on our nose through which we see whatever we look at. It never occurs to us to take them off.

In these passages, Wittgenstein holds up for scrutiny the dual Tractarian demands for definiteness of sense (*PI* 99) and strictness of rules (*PI* 102). The connection should be obvious: If the sense of an expression is determined by the rules that govern its use, then the determinacy/indeterminacy of rules will go hand in hand with determinacy/indeterminacy of sense. I think this pairing of the meaning of an expression with the rules for its application is *common* to Wittgenstein's early and late philosophy. If someone is looking for a deep *similarity* between the two standpoints, this certainly counts as one. Since in his later philosophy Wittgenstein reverses his attitude toward determinacy vs. indeterminacy—shifting, we might say, from the sublime to a mundane conception of rules and meaning—this is also the locus of an equally deep *dissimilarity* between Wittgenstein's early and late philosophy.

So far I have concentrated on Wittgenstein's rejection of the doctrines of the strictness of rules and the corresponding determinacy of meaning manifest in the *Tractatus.* His target, however, is not restricted to the *Tractatus*, for the misunderstandings that he finds there are, he thinks, pervasive in philosophy. I will cite just one more example of Wittgenstein's critique of determinacy: his treatment of philosophical problems concerning personal identity. Philosophical perplexities concerning personal identity arise primarily in two contexts: (a) cases where there are gradual (perhaps imperceptibly small) changes over time, and (b) cases where the changes are bizarre. In the *Blue Book* Wittgenstein considers both, and does so in an elegant and paradigmatically Wittgensteinian manner.

Here, in part, is how he deals with problems generated by *gradual* changes:

> Under what circumstances do we say: "This is the same person whom I saw an hour ago"? Our actual use of the phrase "the same person" and of the name of a person is based on the fact that many characteristics which we use as the criteria for identity coincide in the vast majority of cases. I am as a rule recognized by the appearance of my body. My body changes its appearance only gradually and comparatively little, and likewise my voice, characteristic habits, etc. only change slowly and within a narrow range. We are inclined to use personal names in the way we do, only as a consequence of these facts.

Wittgenstein's treatment of *bizarre and sudden* changes takes the following form:

> Were Dr. Jekyll and Mr. Hyde two persons or were they the same person who merely changed? We can say whichever we like. We are not forced to talk of a double personality. . . . The *ordinary* use of the word "person" is what one might call a composite use suitable under the ordinary circumstances. If . . . these circumstances are changed, the application of the term "person" or "personality" has thereby changed; and if I wish to preserve this term and give it a use analogous to its former use, I am at liberty to choose between many uses, that is, between many different kinds of analogy. One might say in such a case that the term "personality" hasn't got one legitimate heir only. (*B & B*, pp. 61–62)

There are many passages in Wittgenstein's writings where he brings into prominence the fact that the use, hence the meaning, of certain expressions depends on contingent, though reasonably steady, features of the world. It is also a fact that, up to a point, the use of an expression can tolerate imagined changes in these supporting contingencies. Often any one of them—or maybe quite a few of them—can be dropped without the expression losing or significantly shifting its meaning. This can encourage the idea that the meaning of an expression does not rest on a system of contingencies at all, but on something deeper, something more solid and unchanging. It has been thought to be the special calling of philosophy to reveal these deeper, more solid structures. This often leads to complex, subtle, and ingenious efforts to *tease out* a proper analysis of a given concept. Often as not these efforts yield no consensus concerning the correct analysis of the concept. This is certainly true of ex-

aminations of personal identity— from (at least) Locke to the present. Parallel problems arise throughout the philosophical landscape. Wittgenstein is not simply challenging the standpoint of the *Tractatus*, he is calling into question a fundamental conception of the goal of philosophy. I will return to this topic in a coda to this work.

There is a further and more radical difference between the Tractarian treatment of rules and their treatment in Wittgenstein's later philosophy: In the *Tractatus*, the commitment to consistency is so fundamental that it does not have to be explicitly stated. In his later philosophy Wittgenstein abandons this demand by holding that a system of rules can give good service when it is not consistent or free of paradox. Indeed, it can give good service even after the inconsistency has been discovered and can do so without first eliminating the inconsistency. This seemingly outrageous doctrine will be the subject of a later chapter.

On the other hand, we are sometimes led into perplexity, not because we misunderstand or misrepresent *particular* uses of language, but because we ask very general questions concerning language—for example, how is it possible for a word to name an object, or how is it possible to follow a linguistic rule (or any rule)? The response we get from Wittgenstein to such questions is a defactoist response. Here our mistake is not that of confounding one use of language with another; our mistake is seeking a grounding or support where none exists—and none is needed.

The diversity theme and the defactoist theme operate in tandem in Wittgenstein's investigations. This will become evident in the chapters to follow.

Chapter Two

The Conceivability of a Private Language

> On one occasion when I spoke of an unsuccessful philosophical discussion, [Wittgenstein replied] "Perhaps you made the mistake of denying what he said." (John Wisdom, "Ludwig Wittgenstein, 1934–1937")

Wittgenstein's Grammatical Remarks on Private Rule-Following

As indicated in the introduction, I no longer think that Wittgenstein was attempting to establish the strong negative claim that a private language (as specified in *PI* 243) is not possible. I previously based my interpretation of the so-called private language argument on what might be called a *straight* reading of *PI* 202.

> *PI* 202. 'Obeying a rule' is a practice. And to *think* one is obeying a rule is not to obey a rule. Hence it is not possible to obey a rule 'privately': otherwise thinking one was obeying a rule would be the same thing as obeying it.

Given the background assumption that, for Wittgenstein, employing language is a form of rule-governed behavior, it seems to follow at once that the kind of private lan-

guage described in *PI* 243 is not possible. I no longer read *PI* 202 and the propositions that surround it in this way. I now hold that these propositions are *grammatical* reflections on linguistic rule-following intended to show that the notion of a private language lacks coherent content. We think that we understand this notion, but, as grammatical reflection can show, we do not.

What, then, are my reasons for viewing *PI* 202 as a grammatical proposition? Part of the answer, though not all, depends on what, in the previous chapter, I called Wittgenstein's *rich* conception of rule-following. Rules, as Wittgenstein understands them, are regulations governing practices or institutional activities. This, as we saw, is the point of *PI* 199:

> *PI* 199. Is what we call "obeying a rule" something that it would be possible for only *one* man to do, and to do only *once* in his life?

I take it that we are supposed to answer Wittgenstein's rhetorical question, "No, that would not be possible." Rules carry a presumption of stability over time. Having elicited this response, Wittgenstein immediately cautions his reader not to view it in an improper light, telling us that "this is of course a note on the grammar of the expression 'to obey a rule.'" Since I will argue that Wittgenstein's reflections on the possibility of a private language should also be understood as *notes on the grammar of various expressions*, it is important to be clear concerning what Wittgenstein has in mind in speaking of the grammar of an expression.

Wittgenstein, of course, is not using the notion of the grammar of an expression in the narrow sense of its syntax. For Wittgenstein, the grammar of an expression con-

cerns its role or use in the language. According to him, a standing source of philosophical confusion—and this is a recurrent theme in his later writings—is the tendency to transpose a remark concerning the grammar of an expression into a seemingly substantive claim about the things referred to in the expression. To begin with a trivial (and philosophically uninteresting) example, someone, I cannot remember who, used the sentence "A fox does not have a tail; it has a brush" to illustrate the tendency to transpose a linguistic comment into one that seems extra-linguistic. The point of the remark—here I rely on the Oxford English Dictionary—is not that British foxes are tailless, but that in Britain, at least among certain classes, a fox's tail is not called a "tail," but is referred to as a "brush" instead. Wittgenstein, of course, is not interested in shallow examples of this kind. He is concerned with deep misunderstandings that arise when the grammatical characteristics of important classes of judgments are transposed into often misleading substantive counterparts. To cite an example that will be examined later, under certain circumstances we are inclined to say, "Only I can know that I am really in pain, another person can only surmise it" (see *PI* 246). This is—of course!—just false. On many occasions we can know full well that someone else is in pain—there is no surmising about it. Why then, under certain circumstances, are we tempted to say the opposite? And why, when we yield to this temptation, do we have a sense of having said something deep—perhaps having expressed an important philosophical insight? The answer to these questions, in brief, is that there are important differences in the underlying grammar of first-person ascriptions of pain ("I have an ache in my lower back") and third-person ascrip-

tions of pain ("He has an ache in his lower back"). Through a misunderstanding, this grammatical difference is transposed into a difference concerning what we can and cannot know. There will be more on this later.

Wittgenstein draws attention to this tendency to transpose grammatical differences into nongrammatical differences in a variety of ways. Sometimes he has an interlocutor exhibit it. In *PI* 199, he elicits such a response from the reader by asking a rhetorical question: Is obeying a rule "something that it would be possible for only *one* man to do, and to do only *once* in his life?" It is important to see that the negative answer this question calls forth is still out of focus, suggesting that a causal (or, heaven help us, a transcendental) connection is involved. At other times Wittgenstein expresses such misleading transformations in his *own* person and then calls himself to the bar for doing so. Through the use of such devices Wittgenstein exhibits, and sometimes acts out, the emergence of philosophical confusion.

Returning to *PI* 199,Wittgenstein continues the passage with a series of other remarks on rule-following that, properly understood, can only be taken as grammatical notes:

> *PI* 199. It is not possible that there should have been only one occasion on which a report was made, an order given or understood; and so on.—To obey a rule, to make a report, to give an order, to play a game of chess, are *customs* (uses, institutions).

Given this background, how are we to understand the anti-privacy remark in *PI* 202—"It is not possible to obey a rule 'privately' "? The context demands that we not take it straight, but instead treat it as a note on the grammar

of the expression "to obey a rule 'privately.'"[1] If that is right, then any claim to the effect that a private language is *not* possible would be misleadingly expressed, that is, still in need of an explication that neutralizes its capacity to mislead. Perhaps this is why Wittgenstein encloses the word "private" in quotation marks—but I will not press the point.

It will help to compare Wittgenstein's grammatical notes on rule-following with a parallel passage where Wittgenstein says something (this time in his own voice), stops, challenges himself, and then corrects a false impression he has given by declaring that what he has said has grammatical import rather than a seemingly substantive import:

> *PI* 339. Thinking is not an incorporeal process which lends life and sense to speaking, and which it would be possible to detach from speaking, rather as the Devil took the shadow of Schlemiehl from the ground.—But how "not an incorporeal process"? Am I acquainted with incorporeal processes, then, only thinking is not one of them? No; I called the expression "an incorporeal process" to my aid in my embarrassment when I was trying to explain the meaning of the word "thinking" in a primitive way.
>
> One might say "Thinking is an incorporeal process", however, if one were using this to distinguish the grammar of the word "think" from that of, say, the word

[1] Of course, we can think of contexts in which this remark, "It is not possible to obey a rule 'privately'," is intended as a substantive claim and not as a note on the grammar of an expression. The point might be that *nothing* we do can be done in private because God is aware of everything about us, including our most inward thoughts. Wittgenstein is not concerned with invasions of privacy of this kind.

> "eat". Only that makes the difference between the meanings look *too slight*. (It is like saying: numerals are actual, and numbers non-actual, objects.) An unsuitable type of expression is a sure means of remaining in a state of confusion. It as it were bars the way out.

If we substitute the sentence "It is not possible to obey a rule 'privately'" for the sentence "Thinking is not an incorporeal process," then, *mutatis mutandis*, we get the reading of the private language "argument" I am suggesting. It hardly seems plausible to think that Wittgenstein could treat the sentences "Thinking is not an incorporeal process" and "It is not possible to obey a rule 'privately'" differently. Both are ways of speaking that Wittgenstein called to his aid—or better, acted out—when making a grammatical point, as he says, "in a primitive way." Both claims are used in the early ("primitive") stage of his treatment of a philosophical problem.

The Imaginability of a Private Language

I think that understanding Wittgenstein's reflections on private rule-following in *PI* 199–202 as grammatical in character provides the proper perspective for understanding his reflections on a private language found in *PI* 243ff. Here is *PI* 243 in full:

> *PI* 243. A human being can encourage himself, give himself orders, obey, blame and punish himself; he can ask himself a question and answer it. We could even imagine human beings who spoke only in monologue; who accompanied their activities by talking to themselves.—An explorer who watched them and listened

> to their talk might succeed in translating their language into ours. (This would enable him to predict these people's actions correctly, for he also hears them making resolutions and decisions.)
>
> But could we also imagine a language in which a person could write down or give vocal expression to his inner experiences—his feelings, moods, and the rest—for his private use?—Well, can't we do so in our ordinary language?—But that is not what I mean. The individual words of this language are to refer to what can only be known to the person speaking; to his immediate private sensations. So another person cannot understand the language.

The first thing to notice—and this is crucial—is that this passage concerns *imaginability*. We are told that we could imagine people who talk only to themselves. They speak, we might say, an egocentric language. Wittgenstein does not fill in the details of such a language. For example, he doesn't indicate how such people could use language to coordinate their activities—perhaps they eavesdrop on one another. He is content with saying that their talking is connected with their activities, and, given this, an explorer might be able to translate their language into his own. Next Wittgenstein asks whether we can imagine someone keeping a record of his inner experiences for his private use. Perhaps these inner experiences are recorded in a diary that is kept hidden. The entries in the diary might be encrypted. That is certainly imaginable; indeed, for a time, Wittgenstein kept such an encrypted diary himself. With such a diary we are only dealing with language that is *kept* private. With both types of privacy—egocentric privacy and kept privacy—

imaginability is (or at least seems) unproblematic, and we can proceed to ask whether, as a matter of fact, such forms of linguistic privacy actually exist. The answers seem to be that egocentric privacy does not exist and kept privacy does.

Wittgenstein, of course, is not interested in either egocentric privacy or kept privacy; he is concerned with what we might call inherent privacy (über-privacy). Can we imagine someone having a language that no other person can understand because its words refer to things that can only be known to the diarist himself, namely, they refer to his own immediate private sensations? I think that here there is a temptation to say, "Of course, we can imagine this—Wittgenstein just told us quite clearly what we are supposed to imagine." If we yield to this temptation, then this third sort of privacy will be treated in the same way that the earlier two sorts of privacy are treated. We can imagine there being a language that is private in this way; the only question is whether such a language actually exists. Looking at the matter this way, I previously drew the conclusion that, given the kind of creatures we are, a private language of this kind is not a possibility for human beings. I now think that the underlying assumption leading to this conclusion is just wrong. I now see Wittgenstein as challenging, not accepting, the assumption that we can imagine such a language. The challenge, however, does not concern our powers of imagination. Instead, it concerns the grammar of first-person reports of sensations.

These reflections are borne out in the section that immediately follows *PI* 243. *PI* 244 opens with remarks on our *everyday* use of words concerning our own sensations:

> *PI* 244. How do words *refer* to sensations?—There doesn't seem to be any problem here; don't we talk about sensations every day, and give them names?

Wittgenstein next rephrases his question explicitly in terms of the name-relationship: "But how is the connexion between the name and the thing named set up?" This, he tells us, is the same as asking, "How does a human being learn the meaning of the names of sensations?—of the word 'pain' for example?" To answer this question, he describes one way that a child might be taught how to use the word "pain":

> *PI* 244. Here is one possibility: words are connected with the primitive, the natural, expressions of the sensation and used in their place. A child has hurt himself and he cries; and then adults talk to him and teach him exclamations and, later, sentences. They teach the child new pain-behavior.

The remarkable feature of this response is that Wittgenstein does not take the question "How do words *refer* to sensations?" head on. He offers instead an alternative to a straightforwardly referential approach to the way sensation-terms function.[2] The sensation of pain gets connected with language by means of primitive and natural modes of expressions of pain—through moaning, screaming, weeping, rubbing an injured place, and so on. Through training, these primitive and natural expressions of pain are given linguistic shape. In this way the child who feels pain comes to command a concept of pain.

[2] Though Wittgenstein speaks of "one possibility," he never suggests another and writes as if this possibility is at least broadly descriptively correct.

Of course having a pain is not the same thing as having the concept of a pain, any more than having a cup is having the concept of a cup. More to the point, having a pain does not, of itself, *give* one the concept of a pain. Wittgenstein says just this:

> *PI* 384. You learned the *concept* 'pain' when you learned language.

Thus children suffer pains before they have or acquire the concept of pain. Some animals may suffer pain without *ever* acquiring the concept of pain.[3] This, I think, is at least part of what Wittgenstein has in mind when he makes remarks of the following kind:

> *PI* 288. If anyone said "I do not know if what I have got is a pain or something else", we should think something like, he does not know what the English word "pain" means; and we should explain it to him.—How? Perhaps by means of gestures, or by pricking him with a pin and saying: "See, that's what pain is!" This explanation, like any other, he might understand right, wrong, or not at all. And he will shew which he does by his use of the word, in this as in other cases.

Here, I take it, we are supposed to assume that sticking the person with a pin produces the sensation of pain. Even so, this does not close off the possibility of a plurality of reactions the person might have in response to our efforts. He might, for example, take the word "pain" to be the proper name of the pin I used to prick him. For Wittgenstein, that problem, as I argued in the previous chapter, has only a defactoist solution.

[3] That, of course, does not countenance the indiscriminate infliction of pain on animals.

In *PI* 246 Wittgenstein asks, "In what sense are my sensations *private*?" but again he refuses to provide a direct answer. Instead, he cites and dismisses a philosophical account of the privacy of sensations:

> *PI* 246. Well, only I can know whether I am really in pain; another person can only surmise it.—In one way this is wrong, and in another nonsense.

For Wittgenstein, the passage says something wrong because others, quite obviously, can know I am in pain. Many contexts leave no room for doubting this.[4] For Wittgenstein, nonsense appears with the utterance "I know I am in pain." This is more interesting—and perhaps more controversial—for we are strongly inclined to say "Of course I can know that I am in pain, for, after all, *I'm* the one feeling it. What could I know better?"—or something like that.

The reason we may be led to believe that people can know they themselves are in pain, but cannot know this of others, is that we accept the doctrine that sensations are private. Wittgenstein diagnoses this claim as a misconstrual of the grammar of first-person ascriptions of sensations:

[4] Wittgenstein makes parallel remarks concerning ascriptions of pains to others.

> *PI* 303. "I can only *believe* that someone else is in pain, but I *know* it if I am."—Yes: one can make the decision to say "I believe he is in pain" instead of "He is in pain". But that is all.—What looks like an explanation here, or like a statement about a mental process, is in truth an exchange of one expression for another which, while we are doing philosophy, seems the more appropriate one.
>
> Just try—in a real case—to doubt someone else's fear or pain.

> *PI* 248. The proposition "Sensations are private" is comparable to: "One plays patience by oneself".

What Wittgenstein seems to be saying is this: Given (as indicated in *PI* 244) that utterances of the form "I am in pain" function as expressions of one's pain, it is no more possible for you to express my pain than for you to groan my groan or cast my shadow. The claim that "sensations are private" is thus the expression of a grammatical fact distorted in a philosophical haze. Thus, instead of acknowledging that sensations are private in a substantive sense—and not even denying this—Wittgenstein suggests that this sense of privacy is an illusion generated through a misapprehension of the grammar of sentences that express feelings. That Wittgenstein is saying something like this is borne out in *PI* 251:

> *PI* 251. What does it mean when we say: "I can't imagine the opposite of this" or "What would it be like, if it were otherwise?"—For example, when someone has said that my images are private, or that only I myself can know whether I am feeling pain, and similar things.
>
> Of course, here "I can't imagine the opposite" doesn't mean: my powers of imagination are unequal to the task. *These words are a defense against something whose form makes it look like an empirical proposition, but which is really a grammatical one.* [Emphasis added]

From these passages—and indeed from the entire surrounding context—it is clear that Wittgenstein is *not* reasoning in the following way: We perfectly well understand how sensation-terms could refer to something purely private, but a survey of the way such terms are actually used shows that they do not function in this way.

We are tempted to say that sensations are private because we are tempted to give sensation-terms a straightforward referential reading. But, as Wittgenstein tells us,

> *PI* 254. What we 'are tempted to say' in such a case is, of course, not philosophy; but it is its raw material. Thus, for example, what a mathematician is inclined to say about the objectivity and reality of mathematical facts, is not a philosophy of mathematics, but something for philosophical *treatment*.

What this treatment reveals, as I understand it, is that we have no grasp of how sensation-terms could gain their meaning through purely private reference even though, when doing philosophy, we are inclined to think we do.

Private Acts of Ostension

In *PI* 256 Wittgenstein returns to the question posed in *PI* 243: "*How* do I use words to stand for my sensations?"[5] Here it is natural to reply, "I refer to them in just the same way I refer to other things: I give them names and then, using these names, proceed to talk about them." That seems right, for introducing names seems to be something we can do at will. The mistake here, according to Wittgenstein, is to think that ostensive definition provides a direct and unproblematic way of assigning a meaning to a term. Indeed, it is tempting to think that ostensive definitions provide the *fundamental* way in which words get hooked up with objects. On this ap-

[5] Wittgenstein goes on to say, "As we ordinarily do? Then are my words for sensations tied up with my natural expressions of sensation? In that case my language is not a 'private' one." We can hold this move in abeyance for a bit.

proach, the meaning of a name, for example, is just the object correlated with it, a view that Wittgenstein may seem to adopt in the *Tractatus*.[6] In his later writings Wittgenstein rejected this notion. He came to see that, far from being the primitive basis of language, ostensive definition is a high-level activity that presupposes an established linguistic setting. He makes the point in these words:

> *PI* 257. What does it mean to say that he has 'named his pain'?—How has he done this naming of pain?! And whatever he did, what was its purpose?—When one says "He gave a name to his sensation" *one forgets that a great deal of stage-setting in the language is presupposed if the mere act of naming is to make sense*. And when we speak of someone's having given a name to pain, what is presupposed is the existence of the grammar of the word "pain"; it shows the post where the new word is stationed. [Emphasis added]

PI 258 continues this same theme, now applying it specifically to the purely private diary first mentioned in *PI* 243. For reasons that will emerge below, I will initially quote only the first half of this section:

> *PI* 258. Let us imagine the following case. I want to keep a diary about the recurrence of a certain sensation. To this end I associate it with the sign "S" and write this sign in a calendar for every day on which I have the sensation. I will remark first of all that a defi-

[6] In the *Tractatus* Wittgenstein explicitly says, "A name means an object. The object is its meaning" (*TLP* 3.203). But he goes on to say, "Only propositions have sense; only in the nexus of a proposition does a name have meaning" (*TLP* 3.3). So even in the *Tractatus* a bare word/object correlation is not sufficient to give a word the status of a name.

> nition of the sign cannot be formulated.—But still I can give myself a kind of ostensive definition.—How? Can I point to the sensation? Not in the ordinary sense. But I speak, or write the sign down, and at the same time I concentrate my attention on the sensation—and so, as it were, point to it inwardly.—But what is this ceremony for? for that is all it seems to be!

This much of *PI* 258 is a direct continuation of *PI* 257. In producing the private diary, I produce a kind of shadow performance of a standard use of an ostensive definition. I go through the motions, but in a context where these motions make no contact with the mechanism needed to give the sign "S" a particular employment. So far, at least, "S" has not even been given the status of a name. It could as well be a salute, an expression of satisfaction, or whatever. The supposed act of ostensive definition seems no more than an idle ceremony. A bit later, Wittgenstein illustrates the emptiness of this supposed ostensive definition in the following marvelous passage:

> *PI* 268. Why can't my right hand give my left hand money?—My right hand can put it into my left hand. My right hand can write a deed of gift and my left hand a receipt.—But the further practical consequences would not be those of a gift. When the left hand has taken the money from the right, etc., we shall ask: "Well, and what of it?" And the same could be asked if a person had given himself a private definition of a word; etc.

If, however, this act of giving oneself a private definition (in the sense at issue) is empty of content, why doesn't it strike us this way? Well, it seems to be the easiest thing in the world to assign a name to a particular sensation,

and then keep a record of its occurrences. The reason this seems unproblematic is that we tacitly rely on the paraphernalia of our public language in order to provide the surroundings—the stage setting—for giving the expression the status of a name.

> *PI* 261. What reason have we for calling "S" the sign for a *sensation*? For "sensation" is a word of our common language, not of one intelligible to me alone. So the use of this word stands in need of a justification which everybody understands.—And it would not help either to say that it need not be a *sensation;* that when he writes "S", he has *something*—and that is all that can be said. "Has" and "something" also belong to our common language.—So in the end when one is doing philosophy one gets to the point where one would like just to emit an inarticulate sound.

Do reflections of this kind show—are they intended to show—that a private language is not possible? I believe it is wrong to think so. What they are intended to show—and to my mind do show—is that our assumed understanding of the act of assigning names to purely private entities is an illusion. When stripped of the illicit assumption that the standard mechanisms of a public language are in place in this "private" domain, we cannot understand, even to our own satisfaction, such "private" acts of ostensive definition.

Criteria of Correctness

Earlier, I broke off the examination of *PI* 258 in the middle in order to pursue the line of reasoning initiated there. I will pick it up at the point where I broke off:

> *PI* 258. A definition surely serves to establish the meaning of a sign.—Well, that is done precisely by the concentrating of my attention; for in this way I impress on myself the connexion between the sign and the sensation.—But "I impress it on myself" can only mean: this process brings it about that I remember the connexion *right* in the future. But in the present case I have no criterion of correctness. One would like to say: whatever is going to seem right to me is right. And that only means that here we can't talk about "right".

I have delayed a discussion of this part of *PI* 258 until I presented the overall development in the text, because the reference to a "criterion of correctness" may trigger a number of inappropriate associations. For Wittgenstein, criteria are rules for the proper application of terms. These rules—like other rules—can be more or less rigid or determinate. Sometimes, as in mathematics, they are intended to close all loopholes by specifying necessary and sufficient conditions for the application of a term. Sometimes these rules leave loopholes open, yet are serviceable for all that. Furthermore, a sentence gains the status of a rule from the role that is assigned to it (or it naturally acquires) in a given context. Given its status as a rule, it is off-limits to refutation and in that way enjoys an a priori status. It is also worth noting that, just as a proposition can acquire an a priori status by being taken as a rule, a proposition can lose this status by being demoted to an empirical proposition governed by rules. Adopted as a rule, a proposition is normative, policing what is and what is not admissible in the context it governs.

Taken this way, Wittgenstein's insistence that a criterion of correctness is needed to distinguish doing some-

thing from merely seeming to be doing it is substantially the same as his later claim in *PI* 265 that "justification consists in appealing to something independent." A series of specific examples shows what Wittgenstein has in mind by appealing to something independent. It involves looking things up on actual rather than imaginary timetables (see *PI* 265); judging time using clocks that have their hands attached to clockworks rather than simply moving unconnected hands around until their positions strike us as right (see *PI* 266); and not engaging in the foolishness of buying several copies of the morning paper "to assure [ourselves] that what it says is true" (see *PI* 265). These concrete, quite ordinary, ways of checking correctness are what Wittgenstein has in mind when he speaks of "appealing to something independent" as a criterion of correctness. What is indicated by timetables, clocks, and newspapers can serve as criteria of correctness, unless, that is, we have reason to suspect their reliability. Then they can lose this status. They cannot, for obvious reasons, serve as criteria for their own correctness. That is the point of Wittgenstein's remark about the foolishness of buying several copies of the morning paper to assure ourselves that what it says is true.[7] The diarist's use of the sign "S" to stand for a sensation takes place in the total absence of such independent means for checking correctness. The private diarist's activities take place, we might say, in a context where the question of correctness does not come up—no room has been made for incorrectness.[8]

[7] I will return to this phenomenon of turning a criterion back on itself in chapter 3.

[8] Barry Stroud's "Wittgenstein's 'Treatment' of the Quest for 'A Language Which Describes My Inner Experience and Which Only I

PI 270 bears out the claim that the private diarist's activities provide no basis for a contrast between the correct and incorrect application of a term:

> *PI* 270. Let us now imagine a use for the entry of the sign "S" in my diary. I discover that whenever I have a particular sensation a manometer shows that my blood-pressure rises. So I shall be able to say that my blood-pressure is rising without using any apparatus. This is a useful result. And now it seems quite indifferent whether I have recognized the sensation *right* or not. Let us suppose I regularly identify it wrong, it does not matter in the least. And that alone shows that the hypothesis that I make a mistake is mere show. (We as it were turned a knob which looked as if it could be used to turn on some part of the machine; but it was a mere ornament, not connected with the mechanism at all.)

Why, exactly, is it "quite indifferent whether I have recognized the sensation *right* or not"? As a first stab, we may put it this way: Since the operative correlation is between my writing an "S" in my diary and the manometer giving a certain reading, it doesn't matter whether I have "recognized" the sensation correctly or not. But that is not the correct way to put it, for it suggests that mistakes are possible—it just doesn't matter if we make them or not. So Wittgenstein quickly corrects himself by saying,

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Myself Can Understand'" (Stroud 1983) presents a subtle and detailed commentary on *PI* 258, intended to show that Wittgenstein does not invoke the notion of a criterion as the basis for an argument intended to show that a private language (in the specified sense) is impossible. There are some important differences in emphasis between what he says there and what I am saying here, but at bottom they seem to come to much the same thing.

"That alone shows that the hypothesis that I make a mistake is mere show," driving the message home with one of his favorite analogies, a machine part that is mere show because it is unconnected with the rest of the mechanism.

Most of what has been said in this section has been said before—with rings and changes—by me among many others. For example, many of the key moves are found in and elaborated upon in P.M.S. Hacker's *Meaning and Mind*, the third volume of *An Analytic Commentary on the Philosophical Investigations*. There is, however, one aspect to his treatment of these topics that, to my mind, exhibits an intellectualism foreign to Wittgenstein's standpoint. This is his invocation of the notion of *defeasibility* to establish the existence of conceptual connections weaker than those found in analytic propositions but stronger than those found in contingent empirical propositions. I have in mind passages of the following kind:

> [Pain-behaviour] is a *criterion* of pain. It is possible for pain to occur without being manifest, and it is possible for pain-behaviour to be displayed without there being any pain. This grammatical relation, though distinct from entailment, nevertheless allows for certainty, although it is defeasible. (Hacker 1990, p. 243)

In what I take to be feigned self-mockery, Hacker speaks of the "bewildering notion of *a priori* yet defeasible evidence" (p. 560). I confess to finding it bewildering, not only in what it might mean, but why such a notion should be attributed to Wittgenstein. I think that Hacker is attracted to this notion because it seems to produce an argument against skepticism.

> First, a claim to know that another person is in pain . . . cannot be undermined by the fact that the grounds supporting the claim are defeasible, but only by adducing countervailing grounds that defeat them. If the ordinary criteria for someone's being in pain are exemplified in an appropriate context, then the onus of disproof lies with the sceptic, and the logical possibility of defeat is not a defeating condition. Secondly, admitting the possibility of defeating conditions does not mean denying that there are any grammatical limits to defeasibility. In a particular case it may well be that sceptical qualms can be rejected as unintelligible. If someone is thrown into the flames, etc, it makes no sense to say "Maybe he is not in pain but only pretending." (Hacker 1990, pp. 565–66)

It is hard to read these passages in any other way than as presenting something like a linguistic version of a transcendental argument. Take the first claim. What are we to make of the assertion that defeasible claims can be overthrown "only by adducing countervailing grounds that defeat them"? It seems to be a normative claim, but why accept it? One answer, and I think it is Wittgenstein's, is to brush the question aside and say that it is just a fact that human beings do not in general feel challenged by unmotivated defeaters. For many philosophers—including, it seems, Hacker—this retort is not enough. For others—including Dummett and Wright—Wittgenstein shouldn't have made it, and his philosophical value lies elsewhere.

Now consider the second point. In the context as Hacker describes it, it actually could make sense to say

"Maybe he is not in pain but only pretending." The remark could be intended as a very sick joke. Even when the remark is utterly serious, it still makes sense. What we cannot make sense of is not what the person is saying, but why in the world he is saying it. But suppose people constantly went around making wild statements of this kind, what then? The correct answer to this question is not that this is something people can't do, but simply something that they don't do. A signal feature of Wittgenstein's later philosophy is to resist the temptation to say "can't," where we ought only to say "don't."

The notion of defeasibility seems too close to the *intellectualist* notion of a criterion to sit well with what I have called Wittgenstein's defactoism. On the other hand, the phenomenon of propositions shifting back and forth between rules and empirical propositions—or between criteria and symptoms—can do the job that Hacker assigns to defeasibility. It is a theme explicitly found in Wittgenstein's writings, and one that Hacker has examined in detail, and with great insight.

Wittgenstein's Answer

In *PI* 243, Wittgenstein raises this question:

> *PI* 243. [Could we] imagine a language [where] the individual words of this language are to refer to what can only be known to the person speaking; to his immediate private sensations? So another person cannot understand the language.

The immediate and seemingly correct answer to this question is yes. What is Wittgenstein's answer? In fact,

he doesn't say yes and he doesn't say no. If my reading of the text is correct, he neither asserts nor assumes that the answer to this question is yes. In keeping with his repeated warnings about *denying* statements that embody conceptual confusions, he would not answer it with a no either. Instead, he tries to locate the misunderstanding, or patterns of misunderstandings, that give the question itself the false appearance of intelligibility.

II

Wittgenstein on the Philosophy of Mathematics

> An investigation is possible in connexion with mathematics which is entirely analogous to our investigation of psychology. It is just as little a *mathematical* investigation as the other is a psychological one. It will *not* contain calculations, so it is not for example logistic. It might deserve the name of an investigation of the "foundations of mathematics."
> (*PI* II, p. 232)

In this, the closing passage of *Philosophical Investigations*, Wittgenstein draws a parallel—seemingly an exact parallel—between the way one should deal with conceptual confusions in psychology and the way one should deal with them in mathematics. Broadly speaking, the connection is this: In both areas confusions arise from em-

ploying a referential model for understanding a class of expressions where that model is inherently misleading. With respect to mathematical propositions, his primary target is, as it is commonly called, platonism in mathematics.[1] This view, or family of views, admits of wide variation, but, broadly sketched, it embodies the claim that mathematics presents necessary truths concerning distinctively mathematical objects. Wittgenstein's competing conception, briefly stated, is that platonism involves a misunderstanding of mathematical necessity and that platonists' mathematical objects serve no useful purpose in explaining the character or the legitimacy of mathematical activity. But if platonism in mathematics is rejected, aren't we then forced to adopt some version of conventionalism or formalism? Wittgenstein does not think so. He thinks that, with a proper understanding of mathematical propositions, we can see that we are not confronted with a forced choice between platonism on one side and formalism or conventionalism on the other.

Wittgenstein also wishes to deflate what he mockingly refers to as the "mysteries of mathematics." Wittgenstein takes Cantor's infinite hierarchy of ever-increasing infinities as a paradigmatic example of a mysterious realm generated by what he calls "puffed-up proofs."

In addition, Wittgenstein attempts to undermine what he takes to be a pathological fear of inconsistency in mathematics and logic. The desire to establish the con-

[1] I have chosen to write "platonism" with a lowercase "p" in order to avoid giving the impression that I am referring specifically to Plato's views on mathematics. Though Plato was in many ways a platonist in the contemporary understanding of this term, I do not want to get into the difficult business of distinguishing the various forms that platonism can take. Instead, I will rely on context to identify the aspect of platonism under consideration.

sistency of mathematics is one of the driving forces behind the attempt to provide it with a suitable foundation. Wittgenstein thinks that work in the so-called foundations of mathematics, being just another part of mathematics, cannot resolve obsessive fears of inconsistency. Beyond this, and perhaps more outrageously, Wittgenstein also rejects the idea that an inconsistency in a calculus is sufficient to render that calculus useless.

In the next chapters I will discuss these three topics in turn: the status of mathematical expressions, the mysteries of mathematics, and Wittgenstein's seemingly laissez-faire attitude toward inconsistency in logic and mathematics.

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Chapter Three

The Status of Mathematical Expressions

> We are used to saying "2 times 2 is 4", and the verb "is" makes this into a proposition, and apparently establishes a close kinship with everything that we call a 'proposition'. Whereas it is a matter only of a very superficial relationship. (*RFM* I, Appendix 3, 4)

The Reality of Numbers

One of Wittgenstein's central ideas is that philosophers are often misled by taking similarities in syntactical structure as indicators of a similarity in *grammar*—that is, as Wittgenstein uses this notion, as similarities in *use*. Often this assumption is correct, but not always. In the previous chapter we saw that Wittgenstein held that this assumption produces conceptual muddles when applied to first-person ascriptions of mental properties. In the passage cited above, he makes a parallel claim with respect to mathematical expressions. Wittgenstein's suggestion is that we have a tendency, indeed a strong tendency, to treat "2 + 2 = 4" on the model of, say, "Together, gin and dry vermouth make a dry martini." Anyone guided by the comparison will, of course, recognize that the entities involved in the two propositions are fundamentally differ-

ent and their mode of combination, fundamentally different as well. It will also be acknowledged that the two propositions have different methods of verification. Having acknowledged all this, there remains a strong inclination to say that they are still similar in asserting a relationship between entities and are true just in case the entities referred to stand in the indicated relationship. Each statement is true if it squares with or corresponds to the aspect of reality it is about.

In his *Lectures on the Foundations of Mathematics*, Wittgenstein attempts to dislodge this picture with respect to mathematical propositions by invoking a series of comparisons:

> Suppose we said, "A reality corresponds to the word 'two'."—Should we say this or not? It might mean almost anything.
>
> "A reality corresponds to the word 'perhaps'."—Does one, or not? You *might* say so; but nobody would. —Or to "or", or to "and". It is unclear what reality we should say corresponds here. . . .
>
> The point is this. We *can* explain the *use* of the words "two", "three", and so on. But if we were asked to explain what the reality is which corresponds to "two", we should not know what to say.—This? [*He indicated the two fingers.*] But isn't it also six, or four?
>
> We have certain words such that if we were asked, "What is the reality which corresponds?", we should all point to the same thing—for example, "sofa", "green", etc. But "perhaps", "and", "or", "two", "plus" . . . are quite different.
>
> If a man asks, "Does no reality correspond to them? [i.e., to "perhaps," "and," "or," "two," or "plus"] what

> should we say? How should we explain this feeling that there is a reality corresponding to these words too?—He means "Surely we have some use for them." And that is obviously true. (*LFM*, pp. 248–49)

Then with respect to whether a reality corresponds to "2 + 2 = 4," he tells us:

> I don't say: "No reality corresponds."[1]
>
> To say "A reality corresponds to '2 + 2 = 4'" is like saying "A reality corresponds to 'two'." It is like saying a reality corresponds to a rule, which would come to saying: "It is a useful rule, *most* useful—we couldn't do without it for a thousand reasons, not just *one*." (*LFM*, p. 249)[2]

We can first note that in the middle of these passages we find another invocation of the paradox of interpretation: Trying to show the referent of "2" by holding up two fingers (without the proper surroundings) could equally be a way of exhibiting the referent of "6" or "4." But Wittgenstein's chief aim is to try to break the spell of a referential understanding of mathematical expressions by comparing them with expressions that are meaningful but not referential in their employment. His list of expressions is interesting. He begins with the word "perhaps." There may be people who think that "perhaps"

[1] Why *doesn't* Wittgenstein just say, "No reality corresponds to '2 + 2 = 4'"? The answer, I think, is this: To deny that a reality corresponds to this expression suggests that it makes sense to say that a reality does correspond to it, but there simply isn't any such correspondence. Putting it that way simply preserves the conceptual confusion that Wittgenstein is trying to dispel.

[2] James Conant recognizes the importance of this passage and cites it in Conant 1997 (p. 202).

does have a referent, perhaps the hesitancy or indecisiveness of the person using this word. I think this is simply wrong, but I will not try to show this here. The remaining words, "and," "or," "two," and "plus," are logical and mathematical in character. The rejection of a referential understanding of such terms goes back to the *Tractatus*. In proposition 4.0312 of the *Tractatus*, Wittgenstein declared that his "leading idea" was that logical constants are not representatives. In the *Tractatus*, he held a similar view concerning arithmetic terms. This is one area where there is an important continuity between Wittgenstein's earlier and later philosophy.

In the passage just cited, Wittgenstein invites us to look at mathematical expressions from a nonreferential perspective, but he has yet to give us any good reason for adopting this standpoint. He is also aware that the spell of a referentialist view of mathematics is not easily broken, for the misunderstandings that underlie it are, as he tells us, "tenacious" and "hard to get rid of" (*LFM*, p. 15). In the opening lecture of *Lectures on the Foundations of Mathematics*, Wittgenstein describes these misunderstandings and the way he proposes to deal with them:

> What kind of misunderstandings am I talking about? They arise from a tendency to assimilate to each other expressions which have very different functions in the language. We use the word "number" in all sorts of different cases, guided by a certain analogy. We try to talk of very different things by means of the same schema. This is partly a matter of economy; and, like primitive peoples, we are much more inclined to say, "All these things, though looking different, are really the same" than we are to say, "All these things, though looking

> the same, are really different." Hence I will have to stress the differences between things, where ordinarily the similarities are stressed, though this, too, can lead to misunderstandings. (*LFM*, p. 15)

Wittgenstein employs a variety of methods to bring out the underlying differences in the use of syntactically similar-looking expressions. One way, when applicable, is to examine the modes of verification appropriate to various kinds of expressions. Another method, which is more generally applicable, is to examine the training a child receives in developing proficiency in the use of various expressions. Yet another is to construct simple or primitive language games where differences in use are made transparent. Wittgenstein employs all of these techniques—and others—seemingly using the selection principle "Whatever works."

Returning to themes presented in chapter 1, we can first notice how differences in use can be reflected in differences in modes of training. Learning to count is the normal entryway into arithmetic. This usually starts with the rote memorization of an ordered sequence of sounds or marks. Here learning to count is similar to learning one's ABCs. But there are important differences between the two. With counting, at a certain point the student is expected to continue on her own. This doesn't happen with learning the alphabet. The teacher does not say to the student, "Well, now that you have mastered your letters up to M, let's see if you can continue on your own." In learning to count, students master a technique that allows them to develop a series that "lacks the institution of an end," as Wittgenstein in one place puts it (*RFM* II 45).

The difference between numerals and letters becomes more striking when we examine the way in which students are taught to employ these two kinds of symbols. The primary employment of letters is for the phonetic representation of words. (There are other uses as well, including the quasi-mathematical activity of putting a list of words in alphabetical order.) The child is taught the *use* of numerals primarily from learning to count *objects*. There are all sorts of ways that the child can go wrong beyond simply forgetting how to recite the numerals in their proper order. The child might double-count things, skip things, start counting at two rather than one, and so on. However, if this training is successful, as it usually is, after a while the student will get the hang of it and not only be able to recite indefinitely many numerals, but also be able to count up indefinitely many objects. With this the student enters the realm of numbers.

If the child's training proceeds in the standard way, at some point she will be taught *rules* that expedite the employment of numbers. Learning these rules may be accompanied by pithy examples intended to show their legitimacy. Using slashes to represent pencils and parentheses to group them into distinct piles, the justification for "2 + 2 = 4" might look like this:

> (//) (//), taken together, contain the same number of pencils as (////).

The student may see this at once or become satisfied only after counting. Of course, the possibility remains that a student will misunderstand this demonstration or, anyway, find it unconvincing. The paradox of interpretation is always with us. Students may also accept these rules just because they have been pounded into their heads.

We will suppose that the child goes on in the usual way to master the rules for the four basic arithmetic operations (addition, subtraction, multiplication, and division), and then learns to apply them in standard ways in performing practical calculations. Once mastered, these rules will largely fall into the background and operate, we might say, invisibly.

> Suppose that I tell you to multiply 418 by 563. Do you *decide* how to apply the rule for multiplication? No; you just multiply. Probably no rule at all would come into your head. And if one did, no other rule for the application of the first rule would come into your head. *It is not a decision; nor is it an intuition.* (*LFM*, p. 238, emphasis added)

Here Wittgenstein thinks of himself as presenting commonplaces, things that he assumes anyone who has undergone a standard education will at once recognize as true.

So far, at least, nothing seems to present itself as a problem in need of a philosophical solution. Questions like "What is a number?" and "What is the status of mathematical truths?" simply have not come up. This may not seem surprising because, as we might say, we are only dealing with a child's superficial understanding of arithmetic. The deeper, more fundamental, questions have yet to be raised. This is precisely the view that Wittgenstein is attempting to undermine when he says:

> A child has got to the bottom of arithmetic in knowing how to apply numbers, and that's all there is to it. (*LFM*, p. 271)

It is the *and-that's-all-there-is-to-it* clause that is the shocker. The student's mastery of arithmetical techniques

constitutes the "foundation" for her arithmetic activity. This defactoist foundation is not, of course, the kind of foundation that mathematicians and philosophers of mathematics have been seeking.

It might be useful to put our question this way: If arithmetic starts out, as Wittgenstein thinks, as unproblematic, how do problems in the philosophy of mathematics arise at all? Part of his answer is that problems arise concerning the nature of numbers through giving priority to pure mathematics over applied mathematics. On Wittgenstein's approach, arithmetic starts out applied and then is purified by being decoupled from particular applications. On the reverse view that Wittgenstein is challenging, pure mathematics is primary: Applications, though often of great importance in science, technology, and everyday life, are not essential to it. Frege captures the force of this second view using a contrast between the "adjectival" and the "singular" use of numbers. For Frege, when we speak of four apples, we are employing the number four in an adjectival sense. When we say that one plus one equals two, we are not using "one" and "two" adjectivally, but instead we are using them as singular terms referring to numbers. Frege takes the singular use of numerical terms to be primary, their adjectival employment parasitic on their use as singular terms.

> The adjectival use of number is misleading. In arithmetic a number word makes its appearance in the singular as a proper name of an object of this science. . . . The combinations 'each two,' 'all twos' do not occur. (Frege 1997, p. 366)

In *The Foundations of Arithmetic* he remarks:

> We say 'the number 1' and use the definite article to register 1 as an object. This independence manifests itself throughout arithmetic—as, for example, in the equation 1 + 1 = 2. (Frege 1997, p. 106)

Wittgenstein, if I have him right, is reversing these priorities, holding that it is the adjectival employment of numbers outside pure mathematics that is the primary ground of their meaningfulness.[3]

We can see what Wittgenstein is driving at without going into questions in advanced mathematics. The four basic operations of arithmetic will serve our purposes. If we examine these rules without reference to their external employment, our attention can be drawn to what we might call internal relations among them. For example, we can introduce the notion of an even integer by saying that it is an integer that can be divided by 2 without a remainder. An integer is odd if dividing it by 2 leaves a remainder of 1. We can then derive further rules:

> Every integer is either even or odd, never both.
>
> The sum of two even integers is even; of an even and an odd, odd; and of two odd integers, even.

[3] Late in life Frege abandoned the idea that numbers are objects and numerals their names, using language strikingly similar to that which Wittgenstein would later employ.

> One comes to suspect that our way of using language is misleading, that number-words are not proper names of objects at all and words like 'number', 'square number' and the rest are not concept-words; and that consequently a sentence like 'Four is a square number' simply does not express that an object is subsumed under a concept and so just cannot be construed like the sentence 'Sirius is a fixed star'. (Frege 1979, p.263)

> The product of two even integers is even; of an even and an odd, even; and of two odd integers, odd.

Though these expressions are grounded in computational rules, there is something peculiar or awkward about referring to them as *rules*, for they are not themselves used, at least in obvious ways, in performing computations. It seems more natural to say that they express *truths* concerning integers. Wittgenstein makes the point with a more complex example:

> *RFM* II 40. "Fractions cannot be arranged in an order of magnitude."—First and foremost, this sounds extremely interesting and remarkable.
>
> It sounds interesting in a quite different way from, say, a proposition of the differential calculus. The difference, I think, resides in the fact that *such* a proposition is easily associated with an application to physics, whereas *this* proposition belongs simply and solely to mathematics, seems to concern as it were the natural history of mathematical objects themselves.
>
> One would like to say of it e.g.: it introduces us to the mysteries of the mathematical world. *This* is the aspect against which I want to give a warning.

This temptation to accept the existence of purely mathematical objects becomes all but irresistible when we encounter something like Euclid's elegant proof that there is no greatest prime number. We are captured by the image of an unending sequence of primes populating the number series—and we are charmed by it. This, I think, is what Wittgenstein has in mind in the following passage:

> The difficulty in looking at mathematics as we do is to make one particular section—to cut pure mathematics off from its application. It is particularly difficult to know where to make this cut because certain branches of mathematics have been developed in which the charm consists in the fact that pure mathematics looks as though it were applied mathematics—applied to itself. And so we have the business of a mathematical realm. (*LFM*, p. 150)

When mathematics turns pure, it loses contact with ordinary countables: chairs, apples, planets, and so on. This seems to leave us with only two options concerning the status of a mathematical expression: Mathematics consists of a set of rules governing the manipulation of symbols (formalism[4]), or mathematics consists of state-

[4] Formalism (in its various incarnations) gets primitive support from the fact that, when employing computational rules, we are concerned only with the transformation of combinations of symbols. This is particularly striking in solving algebraic equations. We state a problem in an algebraic format, carry out sanctioned transformations that isolate the unknown on one side of the equation, and, *voilà*, the solution *falls out*. Here it is not implausible to think, even if it is wrong to think this, that the symbolic transformations constitute the mathematical component of this activity, and that all the rest is extra-mathematical.

John Stuart Mill put it this way:

> What has led many to believe that reasoning is a mere verbal process, is, that no other theory seemed reconcilable with the nature of the Science of Numbers. For we do not carry any ideas along with us when we use the symbols of arithmetic or of algebra. . . . The ideas which, on the particular occasion, [the symbols of arithmetic and algebra] happen to represent, are banished from the mind during every intermediate part of the process, be-

ments with a distinctly mathematical content of its own (platonism). Wittgenstein dramatizes this seemingly forced conflict between formalism and platonism in the following remarkable passage:

> One asks such a thing as what mathematics is about—and someone replies that it is about numbers. Then someone comes along and says that it is not about numbers but about numerals; for numbers seem very mysterious things. And then it seems that mathematical propositions are about scratches on the blackboard. That must seem ridiculous even to those who hold it, but hold it because there seems to be no way out.—I am trying to show in a very general way how the misunderstanding of supposing a mathematical proposition to be like an experiential proposition leads to the misunderstanding of supposing that a mathematical proposition is about scratches on the blackboard. (*LFM*, p. 112)

Under the influence of a misleading comparison between mathematical propositions and experiential propositions, we seem forced to choose between a theory that is mysterious (platonism) and one that is ridiculous (formalism).[5]

I want to suggest that Wittgenstein's way out of this

> tween beginning, when the premises are translated from things into signs, and the end, when the conclusion is translated back from signs into things. Nothing, then, being in the reasoner's mind but the symbols, what can seem more inadmissible than to contend that the reasoning process has to do with anything more? (Mill 1973, book 2, chapter 6, section 2, p. 254)

[5] As Frege somewhat tendentiously puts it, we face a choice between treating arithmetic as "a game or science." He, of course, opts for the latter (Frege 1997, p. 366).

impasse is utterly simple: Numerals, even when employed in pure mathematics, owe their mathematical significance to their adjectival employment outside pure mathematics. This is precisely what he says:

> *RFM* V 2. I want to say: it is essential to mathematics that its signs are also employed in *mufti*.
>
> It is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics.

A bit later in *Remarks on the Foundations of Mathematics*, he broadens the claim to include all "necessary" propositions:

> *RFM* V 41. Concepts which occur in 'necessary' propositions must also occur and have a meaning in non-necessary ones.

For example, "2" has a meaning in "2 + 2 = 4" in virtue of having a meaning in sentences like "There are two people on the beach." In his *Lectures on the Foundations of Mathematics* he states that "mathematics is not arbitrary, only in this sense, that it has an *obvious* application" (*LFM*, p. 150).[6]

[6] George Berkeley presents a brief narrative of the way numbers and methods of calculation using them came into existence. In virtue of this account, he draws conclusions that are strikingly similar to Wittgenstein's.

> However since there may be some, who, deluded by the specious shew of discovering abstracted verities, waste their time in arithmetical theorems and problems, which have not any use: it will not be amiss, if we more fully consider, and expose the vanity of that pretence; and this will plainly appear, by taking a view of arithmetic in its

Mathematical Necessity

It is a widely held view that $5 \times 5 = 25$ is not only true, but necessarily true. There have been dissenters. Mill was one. He held that the statement that $5 \times 5 = 25$ is a contingent truth with an overwhelmingly high level of empirical confirmation. But even if the view that $5 \times 5 = 25$ is a necessary truth now prevails, it can take various forms, depending on how the notion of necessity is understood. Here, however, I will not canvass these competing theories of mathematical necessity; instead, I will take it as a desideratum that most people who reflect on the matter have a deep inclination to believe that propo-

---

> infancy, and observing what it was that originally put men on the study of that science, and to what scope they directed it. It is natural to think that at first, men, for ease of memory and help of computation, made use of counters, or in writing of single strokes, points or the like, each whereof was made to signify an unit, that is, some one thing of whatever kind they had occasion to reckon. Afterwards they found out the more compendious ways, of making one character stand in place of several strokes, or points. And lastly, the notation of the Arabians or Indians came into use, wherein by the repetition of a few characters or figures, and varying the signification of each figure according to the place it obtains, all numbers may be most aptly expressed.

From this Berkeley concludes:

> In arithmetic therefore we regard not the things but the signs, which nevertheless are not regarded for their own sake, but because they direct us how to act with relation to things, and dispose rightly of them. (*Principles of Human Knowledge*, part 1, sections 121 and 122; Berkeley 1949, p. 96)

sitions of the form $5 \times 5 = 25$ *cannot be false* or, putting it affirmatively, *must be true*. I will largely restrict my discussion to Wittgenstein's account of why people are naturally inclined to speak and think in this way.

Part of what Wittgenstein says about mathematical necessity turns on a simple point: Rules typically tell us what we *must* do, so those who accept the rule $5 \times 5 = 25$ and are operating under it get a *must* straight off. When multiplying 5 by 5, you have to get the result 25. If you get a different result, you have made a mistake. The first thing to say, and not forget, is that this is correct. In calculating, we *do* employ $5 \times 5 = 25$ as a rule or a norm, and normativity brings a *must* with it. The difficulty is that recognizing the normativity of $5 \times 5 = 25$ does not seem to take us very far, for it immediately invites the question, "On what grounds do we, or should we, accept this rule?" One answer is that we accept $5 \times 5 = 25$ as a rule because, taken propositionally, it expresses a necessary truth concerning ideal entities, and that is where the *must* comes from that generates normativity. Wittgenstein notes and rejects this move:

> *RFM* VII 61. What I am saying comes to this, that mathematics is *normative*. But "norm" does not mean the same thing as "ideal".

But if the ideality of its objects is not the source of the normativity of $5 \times 5 = 25$, why do we accept it as a binding rule and as a consequence assign an inexorability to it? Wittgenstein's response to this question involves, I believe, at least four interlocking components: *defactoism*, *mode of acceptance*, *illusions of reflexivity*, and *contextualism*.

Defactoism

One explanation of our acceptance of $5 \times 5 = 25$ as an inexorable rule (or necessary truth) is that it seems possible to produce an entirely convincing proof demonstrating its correctness. For example, I can show someone that 5 times 5 must equal 25 by laying out 5 rows of coins, each containing 5 items.

X X X X X 1.

X X X X X 2.

X X X X X 3.

X X X X X 4.

X X X X X 5.

The person counts up the coins and, sure enough, the result is 25. This demonstration may strike the person as being so perspicuous that 5 times 5 could not equal anything but 25. But maybe not. As we have seen, the paradox of interpretation is always ready at hand to cause mischief. For suppose the person we are dealing with responds by producing the following array of coins:

X X X X X 1.

X X X X X 2.

X X X X X 3.

X X X X X 4.

X

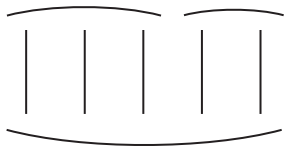
5.

From this she concludes that sometimes, at least, $5 \times 5 = 21$. Has she done what we told her to do? Well, what she has done fits the instructions perfectly, for she has totted up five batches, four horizontally, one vertically, each containing five things. Yet she has not done what we *wanted* her to do; she has yet to master the technique that underlies our use of the expression "5×5." Here we want to convince her that, despite the superficial *similarities* with my performance, she has actually done something quite different. Of course, she might reply that the only difference she can see is that counting the items in our array yields the total 25, whereas when she counts the items in her array she gets 21. But I can hardly invoke this fact to show her that she has not done what we have done, for the whole point of the exercise was to *prove* to her that $5 \times 5 = 25$.

We might try to get around these difficulties by making our instructions more specific, and it's a fact that sometimes making instructions more specific increases a student's chances of getting things right. And it is another fact—this time a conceptual fact examined in detail in chapter 1—that, however specific we make our instructions, there will be some interpretation of what we have said that will support the claim that she has done what we told her to do. Of course, these interpretations will soon seem gratuitous—even mad. But still, from an abstract point of view, anything can be shown to be in conformity with the instructions we have given her. Yet people do, on the whole, follow such instructions correctly, so again we encounter a profoundly Humean theme: a conceptual indeterminacy overbalanced by nothing more than a brute fact of human nature.

This is precisely the line that Wittgenstein takes with respect to the inexorability of the rule 3 + 2 = 5.

RFM I 62.



> Here we have something that looks inexorable—. And yet it can be 'inexorable' only in its consequences! For otherwise it is nothing but a picture.
>
> What does the action at a distance—as it might be called—of this pattern consist in?

RFM I 63. I have read a proof—and now I am convinced.—What if I straightway forgot this conviction?

> For it is a peculiar procedure: I *go through* the proof and then accept its result.—I mean: this is simply what we *do*. This is use and custom among us, or a fact of our natural history.
>
> Or again: it is a fact that I *practically never* get into difficulties in correlating what I have drawn as groups of five.

What we would like to do is "get in back" of this inexorability, to provide it with a legitimizing ground. We have a sense, hard to shake, that Wittgenstein's seemingly psychological inexorability does not supply us with what we are looking for. We want something intellectually more respectable. Yet it seems that whatever we attempt in this direction, the paradox of interpretation emerges to thwart our efforts.

Mode of Acceptance

Wittgenstein imagines someone feeling called upon to make the following proclamation:

> *RFM* VII 67. "If you really follow the rule in multiplying, you *must* all get the same result."

Wittgenstein responds:

> Now if this is only the somewhat hysterical way of putting things that you get in university talk, it need not interest us overmuch.
>
> It is however the expression of an attitude towards the technique of calculation, which comes out everywhere in our life. The emphasis of the *must* corresponds only to the inexorableness of this attitude both to the technique of calculating and to a host of related techniques.

But where does this inexorability come from? Part of the answer is that the techniques of calculating *are given* (perhaps *acquire* is better) this status by being employed in a manner insulated from criticism.

> *RFM* VI 23. The justification of the proposition $25 \times 25 = 625$ is, naturally, that if anyone has been trained in such-and-such a way, then under normal circumstances he gets 625 as the result of multiplying 25 by 25. But the arithmetical proposition does not assert *that*. It is so to speak an empirical proposition hardened into a rule. It stipulates that the rule has been followed only when that is the result of the multiplication. It is thus withdrawn from being checked by experience, but now serves as a paradigm for judging experience.

It is not altogether clear what Wittgenstein is saying here, but I take it that the empirical proposition he is referring to concerns the fact that people who have undergone a certain training almost always come up with 625 when multiplying 25 by 25. Of course, "25 × 25 = 625" doesn't assert this; it makes no reference to people's arithmetical behavior. "25 × 25 = 625" now becomes a rule or measure for judging whether a calculation has been carried out correctly or not. In acquiring this status does it becomes irrefutable? Yes, but only in the sense of gaining a status where refutation has no place, and it gains this status simply by being adopted as a rule, standard, or paradigm.

Here we can imagine someone saying, "Wittgenstein has things backward. It is the recognized necessity of 25 × 25 = 625 that leads us to adopt it as a rule, and not its adoption as a rule that makes it necessary." Wittgenstein sees the force of this primitive complaint, and does not think that it can easily be swept aside:

> *RFM* IV 13. I should like to be able to describe how it comes about that mathematics appears to us now as the natural history of the domain of numbers, now again as a collection of rules.

Later he speaks of the "twofold character of the mathematical proposition—as law and as rule" (*RFM* IV 21). Here, I take it, "law" is used in the sense of a natural law.

Illusions of Reflexivity

Suppose that the "hysterical" member of the university referred to earlier becomes more specific and asks rhetorically, "Isn't it true(!) that 5 × 5 = 25?" It would be

hard to answer this question no, at least straight on, for we do not want to suggest that 5×5 equals something else. Suppose, however, that this same person asked this question, "Isn't it true(!) that $555 \times 555 = 308{,}025$?"—what would we say in response? Unless one is a mathematical genius (or an idiot savant), the natural response would be, "I haven't the faintest idea until I check." The matter might be quickly settled using a calculator. Suppose, however, that no calculator is available and we have to calculate the product the old-fashioned way, that is:

(1)

$$\begin{array}{r} 555 \\ \times\, 555 \\ \hline 2775 \\ 2775 \\ 2775 \\ \hline 308025 \end{array}$$

After checking the calculation for errors, we agree that 555×555 does indeed equal 308,025—or, as we might put it, if you multiply 555 by 555, that is what you get. But this last remark can be viewed in two ways: as a prediction of what someone who has been trained in arithmetic will in fact get when he carries out this calculation, or as a statement of what someone must get when carrying out the calculation *correctly*. The first is an empirical proposition—perhaps a very highly probable empirical proposition—but not a mathematical proposition. The second indicates that the calculation conforms to the arithmetic rules.

In carrying out this calculation we used (among other things) the rule that $5 \times 5 = 25$. Now suppose we use the

same old-fashioned method for calculating the product 5 × 5. It would look like this:

(2)
$$\begin{array}{r} 5 \\ \times\, 5 \\ \hline 25 \end{array}$$

Calculations (1) and (2) seem somehow different. The first showed us that the product of 555 × 555 is 308,025. It was intended to settle that issue, and did. Does the second calculation in the same way settle the issue that the product 5 × 5 is 25? Various responses are possible. One is that there is no issue to resolve. With (2) we finish off the calculation in a flash. We might think of it as a limiting case of performing a calculation. There is, however, this difference: In the long computation, 5 × 5 = 25 was used as a computational rule, and, as a rule, it was not the object or target of the computation. In the second case the rule itself becomes the object or the target of the computation. This activity of turning a rule back on itself is, for Wittgenstein, the source (or at least one source) of our tendency to vacillate between treating 5 × 5 = 25 as a truth concerning relationships among numbers and as a rule of computation governing the employment of numbers. When it is viewed as the outcome of a computation, it is seen as a truth; when it is viewed as a rule governing computations, then it is seen (or taken) normatively. My suggestion—and I think Wittgenstein is saying something very like this—is that this unnoticed switching of perspectives is one important source of the compelling idea that arithmetic is a body of necessary truths.

Wittgenstein adopts a parallel approach in his much-discussed—and, to my mind, often misunderstood—

remarks in *Philosophical Investigations* concerning the standard meter. They occur in a context where Wittgenstein is specifically concerned with philosophers' tendency to posit metaphysical simples or basic elements.

> *PI* 50. One would . . . like to say: existence cannot be attributed to an element, for if it did not *exist*, one could not even name it and so one could say nothing at all of it.

Who would say such a thing? Wittgenstein cites Plato's *Theaetetus* and his own *Tractatus* as examples of texts where such a view is presented. And it is a fact that, under certain circumstances, philosophers can be drawn to such an idea. To show how this can happen, Wittgenstein considers what he calls an "analogous case":

> *PI* 50. There is *one* thing of which one can say neither that it is one metre long, nor that it is not one metre long, and that is the standard metre in Paris.—But this is, of course, not to ascribe any extraordinary property to it, but only to mark its peculiar role in the language-game of measuring with a metre-rule.

Here Wittgenstein is not saying that a metal bar, when used as the standard meter, has no length. He is talking about what it is legitimate to *say* in a context where we recognize that a particular metal bar is being used to fix the length of a meter. Wittgenstein makes a parallel claim concerning a color sample being used as a standard:

> *PI* 50. Let us imagine samples of colour being preserved in Paris like the standard metre. We define: "sepia" means the colour of the standard sepia which is

> there kept hermetically sealed. Then *it will make no sense to say* of this sample either that it is of this colour or that it is not. [Emphasis added][7]

What doesn't make sense is to use something as a standard and simultaneously judge its accordance with that standard. Wittgenstein underscores this point in parallel passages in *Remarks on the Foundations of Mathematics*:

> *RFM* III 36. If I were to see the standard metre in Paris, but were not acquainted with the institution of measuring and its connexion with the standard metre—could I say, that I was acquainted with the concept of the standard metre?

Pretty obviously, the person who sees the object that is the standard meter but is not acquainted "with the institution of measuring and its connection with the standard metre" would not thereby be acquainted with the concept of the standard meter. As we saw earlier, for Wittgenstein, quite generally, the mere acquaintance with an object does not by itself constitute having a concept of that object.

Wittgenstein, however, invokes the function of a standard meter to make a deeper point concerning his own procedures.

> *RFM* III 37. What I always do seems to be—to emphasize a distinction between the determination of a sense and the employment of a sense.

[7] Early in *Philosophical Investigations*, Wittgenstein asks whether the color samples used in the language-game introduced in section 1 are "part of the language" and responds: "It is most natural, and causes least confusion, to reckon the samples among the instruments of the language" (*PI* 16).

The following set of sentences may help to bring out what Wittgenstein is getting at:

(1) Two hundred thousand years ago, there were dogs the size of elephants.

(2) Two hundred thousand years ago, there were elephants the size of dogs.

(3) Two hundred thousand years ago, there were dogs half the size of elephants.

(4) Two hundred thousand years ago, dogs and elephants were the same size.

(5) Two hundred thousand years ago, elephants were the size of elephants.

In (1), (2), and (3), *contemporary* elephants and dogs are invoked as measures of size. In (4), neither dogs nor elephants are set up as standards of size and we are given no indication of the actual size of either of them two hundred thousand years ago. (5) yields two interpretations depending on how *now* and *then* are distributed in the following sentences.

(5′) Two hundred thousand years ago, elephants (then) were the size of elephants (now).

(5″) Two hundred thousand years ago, elephants (then) were the size of elephants (then).

(5′) is a significant claim concerning the size of elephants two hundred thousand years ago. (5″) is true simply in virtue of the reflexivity of the relation *the same size as* and, like (4), gives us no information concerning the actual size of elephants two hundred thousand years ago. (5″) is

the counterpart of saying that the length of the standard meter is one meter.

In our language we sometimes use objects (samples, exemplars, standards, and so on) as instruments in rules that determine sense. Our language also contains noteworthy products that arise from the application of rules. These can take on a normative character as models or paradigms of the proper employment of rules. The point that Wittgenstein is making with these examples is that a dialectical illusion can emerge when a rule is applied to exemplars employed in it. Doubling a rule back on one of its exemplifying instances is, for Wittgenstein, one source, and I am inclined to think one of the most important sources, of the deeply felt attraction of platonism in mathematics.[8]

Contextualism

Earlier I cited a passage from *Lectures on the Foundations of Mathematics*, where Wittgenstein offers an account of what lies behind our tendency to say that a reality corresponds to 2 + 2 = 4:

> It is like saying a reality corresponds to a rule, which would come to saying: "It is a useful rule, *most* useful—we couldn't do without it for a thousand reasons, not just *one*." (*LFM*, p. 249)

I think that Wittgenstein would give the same account of our tendency to say that 2 + 2 = 4 expresses a *necessary truth*, for the rules of mathematics are so widely used and

[8] See Wittgenstein's discussion of the "machine as symbol" in *PI* 193 and my treatment of it in *Wittgenstein* (Fogelin 1987), pp. 156–59. See also *PI* 279: "Imagine someone saying: 'But I know how tall I am!' and laying his hand on top of his head to prove it."

deeply entrenched that it seems inconceivable that we could do without them. Because they are rules that we must have in order to do the many things that we actually do, they become unshakable. In *On Certainty* he puts it this way:

> *OC* 248. One might almost say that these foundation-walls are carried by the whole house.

To sum up, the tendency to treat arithmetical statements as necessary truths is the joint product of a number of mutually reinforcing factors: the brute fact that human beings, after undergoing common training, come to agree on many things; the tendency to assign a protected status to certain expressions and thereby give them the character of rules; the tendency to apply the rules (standards, paradigms) back upon themselves, thus converting normativity into ideality; and the tendency to attribute to particular expressions the force that they derive from their role as part of a system of interlocking expressions. We can think of platonism in mathematics as the product of all these forces acting in concert. Like a neurosis, the misunderstandings on which it rests are multiple and mutually supporting, and for this reason platonism in mathematics admits of no direct and immediate cure.

> *Z* 382. In philosophizing we may not *terminate* a disease of thought. It must run its natural course, and *slow* cure is all important.

Platonists, of course, do not think that they are infected with a "disease of thought" and are not seeking the therapy Wittgenstein has on offer. To quell the lust for mathematical objects, a way has to be found to remove their allure.

Are There Mathematical Objects?

At the start of this chapter I said that two of Wittgenstein's concerns with respect to the foundation of mathematics are the status of mathematical propositions and the status of the objects seemingly referred to by such propositions. For the most part I have concentrated on the first concern: the status of mathematical propositions. Along the way, however, pursuing the first topic inevitably had a bearing on the second. Here I will ask the question directly: How would Wittgenstein answer the question "Are there mathematical objects?" The answer, I think, is that he *wouldn't answer it*—at least not head on. As we saw earlier, he poses the question this way:

> Suppose we said, "A reality corresponds to the word 'two'."—Should we say this or not?

To which he responds:

> It might mean almost anything. (*LFM*, p. 248)

If the question amounts to asking whether we have a *use* for the word "two," then Wittgenstein's answer is yes, there is a reality corresponding to the word "two" (see *LFM*, p. 249). That, however, is not how philosophers usually intend the question. They want to know whether there is a thing or object corresponding to the word "two" in at least the robust way that there are objects corresponding to the word "sofa." However, asking the question this way embodies the assumption that we have a secure grasp of what it is for something to be an object. Of course, we do not. Is a shadow an object? How about an echo, a vector, a pang of remorse, a boundary, phan-

tasm, and so on? We speak *about* all these things and assign properties to them. They are in this broad sense objects of discourse. In order to have significance, saying that there are mathematical objects involves making a comparison, but this comparison can take innumerable forms. We would like to fill in a pattern of the following kind in an interesting way:

> The word "two" has a reality corresponding to it just as the word "X" has a reality corresponding to it.

Wittgenstein suggests:

> The word "two" has a reality corresponding to it just as the word "perhaps" has reality corresponding to it—that is, it has a use.

This is clearly not what the platonist is looking for. How about:

> The word "two" has a reality corresponding to it just as the word "sofa" has reality corresponding to it—that is, its referent.

The platonist would not like this either because a sofa is not the sort of referent he has in mind. All the same, for a platonist, it is an improvement over Wittgenstein's efforts. But even if we grant that the word "two" refers to a number in the same robust sense that the word "sofa" can refer to an article of furniture, we are still left in the dark concerning how such mathematical entities inform mathematical practice. Who needs them, and why?

Geometry seems to provide a fertile field for platonism. Are there geometrical objects? Well, circles are objects and geometry studies circles. Manhole covers are circular, but geometers, of course, have no interest in

them. They do, however, concern themselves with circles that appear in diagrams. We can think of geometrical diagrams as mathematical objects in the modest sense that they are objects that find employment when doing mathematics. The diagrams found in various proofs of the Pythagorean theorem provide a rich source of examples of objects that are mathematical in this respect.

A number of different stories can be told about how diagrams gain their mathematical status. I will consider only two. The first is Plato's story—and here I am speaking about Plato's own theory, not generic platonism. Plato held that a diagram is an image or reflection of an ideal object. He relied on the following analogy: Just as the diagram, taken as a physical object, can have an imperfect reflection, say, in water, the diagram can itself be viewed as an imperfect reflection of an ideal (perfect) structure. Here's how Plato puts it:

> Although [mathematicians] use visible figures and make claims about them, their thought isn't directed to them but to those other things that they are like. They make their claims for the sake of [the] square itself and the diagonal itself, not the diagonal they draw, and similarly with the others. These figures that they make and draw, of which shadows and reflections in water are images, they now in turn use as images, in seeking to see those others themselves that one cannot see except by means of thought.[9]

On this account, a physical diagram gains its mathematical significance through its relationship to this other,

[9] Plato 1997, p. 1131, 510e.

nonphysical, more genuine, mathematical object. Theorems of geometry express truths concerning these ideal objects.[10]

Wittgenstein's reflections on mathematics suggest an alternative view: that a diagram gains its mathematical status in virtue of its mode of employment, not in virtue of a relationship to an independent mathematical object. This passage comes from *Philosophical Grammar*:

> Often we . . . derive geometrical propositions from a drawing, a representation (or a model). But what is the role of the model in such a case? It has the role of a sign, a sign employed in a particular game. . . . And it is this sign, (which has the identity proper to a sign) that we take to be the cube in which the geometrical laws are already laid up. (They are no more laid up there than the disposition to be used in a certain way is laid up in the chessman which is the king).
>
> In philosophy one is constantly tempted to invent a mythology of symbolism or of psychology, instead of simply saying what we know. (*PG* I, 18)

The geometrical cube is no more than a mythical counterpart of a drawing being used as a sign governed by mathematical rules.

[10] Elsewhere I have defended the heterodox view that Plato's Divided Line employs a mathematical diagram to explain the nature of mathematical diagrams. Letting MDs stand for mathematical diagrams, the basic analogy is this:

> Images of MDs are to MDs as MDs are to Mathematical Forms

These relationships, I argue, are reflected in the proportional structure of the diagram itself. For the textual defense of this reading, see Fogelin 1971.

In his *Lectures on the Foundations of Mathematics* Wittgenstein makes the same point in more vivid detail:

> Frege, who was a great thinker, said that although it is said in Euclid that a straight line *can* be drawn between any two points, in fact, the line already exists even if no one has drawn it. The idea is that there is a realm of geometry in which the geometrical entities exist. What in the ordinary world we call a possibility is in the geometrical world a reality. In Euclidean heaven two points are already connected. This is a most important idea: the idea of possibility as a different kind of reality; and we might call it a shadow of reality. (*LFM*, pp. 144–45)

I do not know whether he intends this or not, but in speaking of geometrical objects as shadows, Wittgenstein inverts the basic imagery of Plato's Allegory of the Cave. In any case, Wittgenstein responds to this picture by invoking another variant of the paradox of interpretation.

> We multiply 25×25 and get 625. But in the mathematical realm 25×25 is *already* 625. The immediate [objection] is: then it's also 624, or 623, or any damn thing—for any mathematical system you like. . . . There would be an infinity of shadowy worlds. Then the whole utility of this breaks down, because we don't know which one of them we're talking about. . . . You want to make an investigation, but no investigation will do, because there is always freedom to go into another world. (*LFM*, pp. 145)

Having produced what may seem to be a knockdown argument against Frege's platonism, Wittgenstein ends on what may seem to be a curious note:

> This doesn't at all destroy Frege's argument; it merely shows there is something fishy. (*LFM*, p. 145)

Why don't Wittgenstein's reflections destroy Frege's argument? The answer, I think, is this: Wittgenstein has not attempted to show that there is no mathematical world where 25×25 is already 625. That would involve the mistake of denying what Frege said, thereby granting the intelligibility of Frege's views. I read irony into the claim that he has *merely* shown Frege's views to be fishy. If they are fishy, they are not forced upon us.

For a platonist, mathematical proofs establish truths concerning ideal mathematical objects. For Wittgenstein they are, or could also be viewed as, constructions within a system of mathematical rules. Which of these views is more attractive? If this is a factual question concerning which view has attracted more followers, historically platonism (in one form or another) wins hands down. For someone working *within* a system of mathematical rules, the proof of a new theorem has the phenomenological feel of a discovery of a mathematical fact—something that was already there, not something invented under the governance of rules. Shift the perspective to a descriptive mode *about* the employment of rules, and the situation is reversed: mathematicians are seen as inventors, not discoverers. Wittgenstein's method is to show that choices are open, even when something, for example, platonism, seems forced upon us. An elegant example of this method at work is exhibited in Wittgenstein's treatment of Cantor's theory of transfinite numbers. This is the topic of the next chapter.

Chapter Four

*Wittgenstein on the Mysteries of Mathematics**

> One would like to say [that Cantor] introduces us to the mysteries of the mathematical world. *This* is the aspect against which I want to give a warning. (*RFM* II 40)

Cantorian Infinities

The previous chapter concerned what might be called *global* problems that arise in the philosophy of mathematics, problems, for example, concerning the status of numbers and the status of arithmetic propositions. This chapter will deal with a philosophical problem that arises with respect to a specific result within mathematics: Cantor's discovery or introduction of transfinite cardinals.[1]

Here, as before, I will take Wittgenstein's methodological remarks at face value. I will assume that he means it when he says such things as:

> "It will be most important not to interfere with the mathematicians." (*LFM*, p. 13)

* I have profited greatly from the comments and criticism this material received when presented in various forms at the University of Bologna, Dartmouth College, and the University of Girona. Remarks by Giovanna Corsi, Walter Sinnott-Armstrong, and Sam Levey led to significant revisions.

[1] For now I will leave it open whether *discovery* or *introduction* is the more proper way of characterizing Cantor's achievement.

Or, more fully:

> *PI* 124. Philosophy may in no way interfere with the actual use of language; it can in the end only describe it.
>
> For it cannot give it any foundation either.
>
> It leaves everything as it is.
>
> It also leaves mathematics as it is, and no mathematical discovery can advance it.

I will try to show in some detail how Wittgenstein's reflections on Cantor's treatment of transfinite cardinal numbers conform to these restrictions and are easily misunderstood if this is not recognized. The central idea is this: Wittgenstein is not concerned with the calculations, computations, and derivations of mathematicians. They stand, or do not, on their own mathematical feet. His target is the conceptual misunderstandings that are superadded to legitimate mathematical activity.

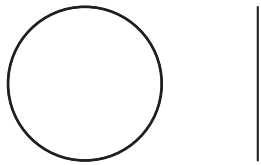
There are several sources of Wittgenstein's views concerning Cantor's treatment of transfinite numbers. One is found in part 2 of his *Remarks on the Foundations of Mathematics*. This material was written by Wittgenstein, the editors tell us, in 1938. Another is found in notes taken by those attending Wittgenstein's lectures on the philosophy of mathematics at Cambridge in 1939. They were compiled and edited by Cora Diamond and published under the title *Wittgenstein's Lectures on the Foundations of Mathematics*.[2] Relying mostly on these texts, I will try to show how Wittgenstein's reservations concerning transfinite numbers are grounded in the broader aspects of his philosophical standpoint. This account will pass through three stages, reflecting what I take to be a pro-

[2] There are other interesting sources as well, for example, in *Philosophical Grammar* II 39–45.

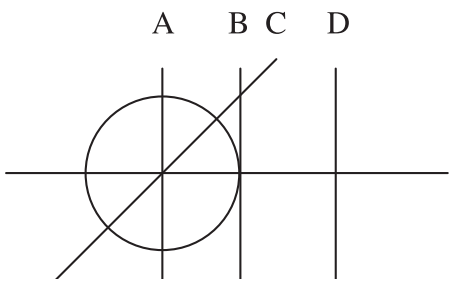
gressive deepening of Wittgenstein's thoughts concerning transfinite numbers.

Stage 1

Near the start of his *Lectures on the Foundations of Mathematics*, Wittgenstein presents an arresting example of what he calls a mathematical misunderstanding.

> There is a kind of misunderstanding which has a kind of charm:
>
> 
>
> "The line cuts the circle but in imaginary points." This has a certain charm, now only for schoolboys and not for those whose whole work is mathematical. (*LFM*, p. 16)

Wittgenstein does not explain what he has in mind in saying that this line is able to cut the circle at imaginary points, and, as far as I know, no commentator has gone to the trouble of spelling this out. I think what Wittgenstein almost certainly has in mind is a curious result that arises in analytic geometry. It can be illustrated by the following diagram:



Here we have a circle with its center at the origin, with a radius of 1, with lines *A*, *B*, and *D* perpendicularly crossing the x-axis at 0, 1, and 2 respectively. *C* cuts through the origin at a 45° angle in the way indicated. The equation for a circle is: $x^2 + y^2 = r^2$. Substituting the appropriate values for each of these lines, we get these results:

Equations	Intersections
With *A*, $x = 0$, so $y = \pm 1$	(0,1), (0,–1)
With *B*, $x = 1$, so $y = 0$	(1,0)
With *C*, $x = y$, so both $\times$ and y must = $\pm\sqrt{1/2}$	$(\sqrt{1/2},\sqrt{1/2})$, $(-\sqrt{1/2},-\sqrt{1/2})$
With *D*, $x = 2$, so, doing the algebra, we get	$(2,\sqrt{-3})$, $(2,-\sqrt{-3})$

A, *B*, and *C* are unproblematic in their relationship to the circle. *A* and *C* cut the circle in a straightforward way; *B* does not cut the circle, but intersects it as a tangent. But the algebraic solution, *D*, which looks like a clear miss, also has an algebraically determined intersection at two imaginary points, namely, (2,–3) and $(2,-\sqrt{-3})$.

Wittgenstein's analysis of this example is interesting because it provides a model for his later remarks on the introduction of transfinite cardinals:

> "Cut" has the ordinary meaning: ∅ . But we prove that a line always cuts a circle—even when it doesn't. Here we use the word "cut" in a way it was not used before. We call both "cutting"—and add a certain clause: "cutting in imaginary points, as well as real points". Such a clause stresses a likeness.—This is an example of the assimilation to each other of two expressions. (*LFM*, p. 16)

The likeness between what we might call real cuts and imaginary cuts consists in the fact that they are calculated using precisely the same algebraic procedure. And notice that the imaginary cuts occur in two quite specific imaginary places. So even if we are dealing with an imaginary structure, it is a determinate and highly articulated structure. This may lead us to think these imaginary cuts occur in a space or dimension that is real in its own special way, and with this, we may think that we have been introduced into a new and mysterious mathematical realm.

Wittgenstein is not suggesting that mathematicians in fact introduced imaginary numbers because they were captivated or charmed by the paradoxical notion of imaginary cuts. He denies this, telling us that calculations employing imaginary numbers "have their use not in charm but in their practical consequences" (*LFM*, p. 16).[3] But suppose, contrary to fact, that it was the charm of imaginary cuts, not practical consequences, that led to the introduction of imaginary numbers. That, Wittgenstein tells us, would alter the status of imaginary numbers in a fundamental way:

> It is quite different if the main or sole interest is this charm—if the whole interest is showing that a line does cut when it doesn't, which sets the whole mind in a whirl, and gives the pleasant feeling of paradox. (*LFM*, p. 16)

[3] In Lecture 23 Wittgenstein remarks that, "by surrounding $\sqrt{-1}$ by talk about vectors, it sounds quite natural to talk of a thing whose square is -1. That which at first seemed out of the question, if you surround it by the right kind of intermediate cases, becomes the most natural thing possible" (*LFM*, p. 226). For example, multiplying a vector by $\sqrt{-1}$ rotates it by 90 degrees counterclockwise—a perfectly understandable, as well as useful, mathematical operation.

This sets the stage for an explicit comparison between the illicit way imaginary numbers *might* have been introduced and the way in which "numbers bigger than infinity" (higher orders of infinities) were *actually* introduced:

> If you can show there are numbers bigger than the infinite, your head whirls. This may be the chief reason [they were] invented. (*LFM*, p. 16)

Except for minor hedging, this seems to suggest that "numbers bigger than infinity" were introduced because of the giddiness reflecting upon them produces.[4]

It will be useful to work out this comparison in some detail. The key idea is that the discovery of a likeness between two notions can lead one to assimilate them in a way that runs counter to obvious differences. It is this clash between similarity and difference that can put the mind in a whirl. We know what it is for a line to cut or intersect a circle. We then are shown that the points where the line cuts or intersects the circle can be determined using a standard algebraic procedure. We now have two ways of representing a cut or intersection: one geometric, the other algebraic. Things fall apart, however, when we consider imaginary solutions to the algebraic equations, for now the algebraic solutions have no geometric counterparts on the real plane. To preserve parity, we could—though nobody does this—introduce the notion of imaginary cuts, that is, cuts that occur in

[4] As we shall see, Wittgenstein's critique is not limited to higher-order infinities, as his casual phrase "numbers bigger than the infinite" may suggest. His target includes all transfinite cardinals, including the starter transfinite cardinal $\aleph_0$.

the domain of complex numbers, not simply in the domain of real numbers.[5]

What would the counterpart story for transfinite cardinals look like? Retelling a familiar tale, suppose a room contains both women and men. There are two ways that we can determine whether there are the same number of men as women in the room. The first is to count up the men and count up the women and then see if we get the same result, say, 27. This is one way of showing that the two groups are equinumerous or have the same cardinality. There is, however, another way of showing that these two groups have the same cardinality without determining what that cardinality is. We note that the room contains only dancing couples, each made up of a woman and a man. Establishing this one-to-one correspondence, as it is called, shows that there are *as many* men as women in the room.[6] Extending this technique in familiar ways, we could also establish, if it is the case, that there are more men in the room than women or that there are more women in the room than men—again doing so without counting.

Primed by these homely and unproblematic examples, we extend the use of one-to-one correspondence in a number of ways. From applying it to familiar collections such as groups of people, we now apply it to abstract sets.

[5] Sam Levey remarked that this might be taken to show that there is a lot more to circles than we originally thought.

[6] The notion of one-to-one correspondence, though not the phrase, is found in Hume's *Treatise of Human Nature*: "When two numbers are so combin'd, as that the one has always an unite answering to every unite of the other, we pronounce them equal." He invokes it to explain the advantage that arithmetic has over geometry with respect to certainty (Hume 2000, p. 51 [book 1, part 3, section 1]).

In daily life we speak of various kinds of sets: sets of dueling pistols, sets of golf clubs, sets of dinnerware, and so on; but Cantor employed the notion of a set in a different way—one not altogether easy to grasp. He offered various formulations of what he meant by a set. For example, in a letter written in 1899, Cantor defined a set this way:

> When . . . the totality of elements of a multiplicity can be thought without contradiction as 'being together', so that their collection into '*one* thing' is possible, I call it a *consistent multiplicity* or a *set*.[7]

Despite superficial similarities, the notion of a set used this way is fundamentally (categorially) different from the notion of a set when we speak, say, of a set of dishes. Provided that they have not been destroyed or dispersed, a set of dishes has a location, a total weight, a monetary value, and so on. In contrast, a set in Cantor's sense whose elements are dishes has no such features. Though the elements of a set may be concrete entities, for Cantor (and for most set-theoreticians) the set itself is not. The things that are members of a set can be almost, though not quite, anything. They can be concrete things (items in a set of dishes) or abstract things (odd numbers). A set can even contain a mixture of entities of each sort. Finally, the sets can be infinite, and not simply finite. We have now traveled a long way from the simple example that introduced us to the notion of one-to-one correspondence, yet despite these changes we continue to use the notion of one-to-one correspondence as the criterion for sameness of cardinality. Combined with flashes of genius,

[7] Cited in Hallett 1984, p. 34.

this is essentially all that is needed to produce a shower of Cantorian results.

I. Though it may seem that there are twice as many integers as even integers, given that they can be put into one-to-one correspondence by what we might call the simple pairing procedure, they are actually equinumerous or have the same cardinality.

II. The rational numbers are equinumerous with the integers as well, for they can be put into one-to-one correspondence with the integers, using what we can call the serpentine procedure.

III. Though the set of rational numbers can be put into one-to-one correspondence with a subset of the real numbers, Cantor's diagonal argument shows that the set of real numbers cannot be put into one-to-one correspondence with the set of rational numbers. The rational numbers can, however, be mapped into the real numbers. Thus the set of real numbers has a higher cardinality than the set of rational numbers.

IV. Beyond this, using the power-set argument, it is possible to prove the existence of an infinite sequence of infinite sets, each with a higher cardinality than its predecessor.[8]

We have now entered what David Hilbert called "the paradise that Cantor has created" (cited in *LFM*, p. 103).

[8] I have given a brief statement of these arguments—together with their standard diagrams—in an appendix to this chapter titled "CantorLand."

It should be obvious why Wittgenstein thinks that this way of introducing transfinite cardinals parallels the charm-driven way of introducing imaginary cuts. I'll spell it out anyway. Both begin with unproblematic notions:

Intersecting lines (cuts) on the real plane	Groups of commonplace objects

Each employs a mathematical method that seems unproblematic in its original domain of application:

Algebraic calculation	One-to-one correspondence

These procedures are then extended to a wider, actually different, domain:

Imaginary points, as well as real points	Infinite sets, as well as finite sets

The upshot is the apparent discovery of a new (and wondrous) realm of mathematical entities:

Imaginary cuts executed at a distance	An infinite hierarchy of transfinite cardinals

In each of these developments, there are components that seem mathematically unproblematic. The algebraic computations (including the derivation of imaginary roots) are unproblematic. The use of one-to-one correspondence as a criterion of equal cardinality for finite sets also seems unproblematic. Nor does Wittgenstein have, at least at first, complaints against Cantor's diagonal argument. He thinks that the diagonal argument shows just what it shows: namely, that the set of real numbers cannot be ordered in a way that the rational numbers can be ordered. However, the diagonal argument yields "num-

bers bigger than infinity" only when combined with the *decision* to extend the use of one-to-one correspondence as a criterion of the same cardinality from finite sets to infinite sets.[9] Nothing forces this decision, and without it the Cantorian results do not emerge. Wittgenstein puts it this way in *Remarks on the Foundations of Mathematics*:

> *RFM* II 22. If it were said: "Consideration of the diagonal procedure shews you that the *concept* 'real number' has much less analogy with the concept 'cardinal number' than we, being misled by certain analogies, are inclined to believe", that would have a good and honest sense. But just the *opposite* happens: one pretends to compare the 'set' of real numbers in magnitude with that of cardinal numbers. The difference in kind between the two conceptions is represented, by a skew form of expression, as difference of extension. I believe, and hope, that a future generation will laugh at this hocus pocus.

For Wittgenstein, the hocus-pocus involves transforming a *qualitative* difference—namely, that certain numbers do not admit of a type of ordering that others do—into what seems to be a stunning claim about comparative *sizes*. In the previous section, Wittgenstein issues a warning against inflating proofs in this way.

> *RFM* II 21. Our suspicion ought always to be aroused when a proof proves more than its means allow it.

[9] This decision reflects what Michael Hallett calls Cantor's principle of finitism: "The transfinite is on a par with finite and mathematically is to be treated as far as possible like the finite" (Hallet 1984, p. 7).

> Something of this sort might be called 'a puffed-up proof'.

Wittgenstein was hardly alone in challenging Cantor's notion of an infinite hierarchy of actual infinities.[10] Still, Wittgenstein's treatment of this topic is emblematic of his way of dealing with a wide range of what he takes to be philosophical misunderstandings, and for this reason still worth exploring. Wittgenstein is not—and this is important—*denying* the existence of transfinite cardinals, for to deny their existence is to acknowledge their intelligibility. This, I think, is the reason he insists, on a number of occasions, that he is not a finitist.[11] We might say that he is a deflationist with respect to transfinite cardinals, but that label can trigger inappropriate associations as well. We might better call him a descriptivist who says things like this:

> I would say, "I wouldn't dream of trying to drive anyone out of this paradise [which Cantor has created]." I would try to do something quite different: I would try to show you that it is not a paradise—so that you'll leave of your own accord. I would say, "You're welcome to this; just look about you." (*LFM*, p. 103)

[10] For example, David Hilbert sought to find a way to avoid reliance on infinities that would mirror Weierstrass's treatment of infinitesimals: "Just as in the limit processes of the infinite calculus, the infinite in the sense of the infinitely large and the infinitely small proved to be merely a figure of speech, so too we must realize that the infinite in the sense of an infinite totality, where we still find it used in deductive methods, is an illusion." He was not successful in this. (The passage is from "Of the Infinite," reprinted in Benacerraf and Putnam 1983, p. 84.)

[11] See, for example, *LFM*, pp. 111 and 141, and *RFM* II 61.

And this:

> I am *not* saying transfinite propositions are *false*, but that the wrong pictures go with them. And when you see this the result may be that you lose your interest. It may have enormous consequences but not mathematical consequences, not the consequences which the finitists expect. (*LFM*, p. 141)

I have, on occasion, presented these Wittgensteinian reflections on transfinite cardinals to mathematicians who specialize in such matters. For the most part, they have dismissed them with amused condescension. "Are we to believe," they more or less say, "that the extraordinarily deep, complex, and sometimes beautiful results that have been found in transfinite mathematics amount only to so much piffle?" Wittgenstein would reply, I suppose, that he has no complaints against the proofs themselves—they prove what they prove, and some of these proofs may be extraordinarily deep, complex, and beautiful—he is only against the gas that inflates them. But saying that Wittgenstein is *against* the inflation of these proofs may be wrong too, for that suggests he has an opposing opinion on this matter, and this is something he explicitly denies.

> One of the greatest difficulties I find in explaining what I mean is this: You are inclined to put our difference in one way, as a difference of *opinion*. But I am not trying to persuade you to change your opinion. I am only trying to recommend a certain sort of investigation. If there is an opinion involved, my only opinion is that this sort of investigation is immensely important, and very much *against the grain* of some of you. (*LFM*, p. 103)

Wittgenstein is right about one thing: The kind of investigation he recommends runs against the grain of many mathematicians and many philosophers of mathematics. Let's make a fresh start by looking at the landscape from a different standpoint.

Stage 2

From what has been said thus far, it may seem that Wittgenstein is content with the notion of an infinite set and is concerned only with challenging the puffing up of Cantor's diagonal argument into a proof of the existence of "numbers bigger than infinity." In fact, his concerns begin with the first, smallest, order of infinity, $\aleph_0$. The following exchange between Wittgenstein and Alan Turing occurs in Lecture 2.

> *Wittgenstein*: We have all been taught a technique of counting in Arabic numerals. We have all of us learned to count—we have learned to construct one numeral after another. Now how many numerals have you learned to write down?
>
> *Turing*: Well, if I were not here, I should say $\aleph_0$. . . .
>
> *Wittgenstein*: Now should we say, "How wonderful—to learn $\aleph_0$ numerals, and in so short a time! How clever we are!"?—Well, let us ask, "How did we learn to write $\aleph_0$ numerals?" . . .
>
> I did not ask, "How many numerals *are* there?" This is immensely important. I asked a question about a human being, namely, "How many numerals did you learn to write down?" Turing answered "$\aleph_0$" and I agreed. In agreeing, I meant that this is the way in which the number $\aleph_0$ is used.
>
> It does not mean that Turing has learned to write

> down an enormous number. $\aleph_0$ is not an enormous number.
>
> The number of numerals Turing has written down is probably enormous. But that is irrelevant; the question I asked is quite different. To say that one has written down an enormous number of numerals is perfectly sensible, but to say that one has written down $\aleph_0$ numerals is nonsense. (*LFM*, pp. 31–32)

Now why does Wittgenstein *agree* with Turing in saying that the student has learned to write down $\aleph_0$ numerals, but that it is nonsense (not just false) to say "that one has written down $\aleph_0$ numerals"? How, someone might ask, can it be sensible to say that one has learned to do something, yet be nonsense to say that one has done it? Wittgenstein's answer comes in a difficult passage in Lecture 17.

> For instance, I tell you, "Write down the first few terms of an $\aleph_0$"; and then you will perhaps write down "1, 2, 3, 4, . . ." or "1, 4, 9, 16, . . ."
>
> Or: "Go on building different streets as far as you can. But one thing: number the houses in each one with a different $\aleph_0$." This is all right.
>
> But *not* "There are $\aleph_0$ trees in this row." (*LFM*, pp. 169–70)

First, there is something peculiar or nonstandard about Wittgenstein's use of the phrase "an $\aleph_0$." His examples, however, indicate that what he has in mind is an $\aleph_0$ *series*. We might, as is often done, number houses on one side of a street using the $\aleph_0$ series of ascending even numbers, houses on the other side using the $\aleph_0$ series of ascending odd numbers. This is fine. What is not fine, according to Wittgenstein, is to invoke the image of $\aleph_0$ addresses al-

ready there, waiting to be assigned to houses as they are built.

Wittgenstein suggests that this imagery of an endless row of integers already in existence is evoked by the series of dots that we append to the sequence of numerals:

> The dots introduce a certain picture: of numbers *trailing off* into the distance too far for one to see. And a great deal is achieved if we use a different sign. Suppose that instead of dots we write Δ, then "1, 2, 3, 4, Δ" is less misleading. (*LFM*, p. 170)

Is Wittgenstein suggesting that people (very intelligent people) are taken in by the imagery of a series of dots? Many will find this wholly unconvincing. He is more persuasive when he speaks of the expression "and so on," which we sometimes use instead of dots.

> There are two ways of using the expression "and so on". If I say, "The alphabet is A, B, C, D, and so on", then "and so on" is an abbreviation. But if I say, "The cardinals are 1, 2, 3, 4, and so on", then it is not. (*LFM*, pp. 170–71)[12]

It is, of course, true that a child, when appropriately trained, can recite vastly more distinct Arabic numerals than distinct letters, but, for Wittgenstein, concentrating on size puts the emphasis in the wrong place. In his *Remarks on the Foundations of Mathematics*, he makes the point this way:

[12] Wittgenstein adds that the British mathematician G. H. Hardy "speaks as though it were an abbreviation." Wittgenstein is referring to "Mathematical Proof" (Hardy 1929), where Hardy presented a version of mathematical realism of the kind that Wittgenstein was challenging.

> *RFM* II 45. To say that a technique is unlimited does *not* mean that it goes on without ever stopping—that it increases immeasurably; but that it lacks the institution of the end, that it is not finished off.

Once more we might say, "A difference in kind between the two conceptions [this time the difference between terminating and nonterminating procedures] is represented . . . as a [vast] difference of extension." Charmed by this picture, we may succumb to it.

Stage 3

A central part of Wittgenstein's attempt to provide an alternative to platonism in mathematics is his suggestion that we view mathematical propositions as rules. As we saw, this brings into play the paradox of interpretation and the defactoist response to it. In what I take to be Wittgenstein's deepest critique of the Cantorian hierarchy, he applies the paradox of interpretation to the use of one-to-one correspondences for determining the cardinality of infinite sets.

> A one-one correlation is nothing but a picture.
>
> X————————X
> X————————X
> X————————X
>
> And you can use this in all sorts of ways. . . .
>
> It is said to be a consequence of Russell's theory that there are as many even numbers as cardinal numbers, because to every cardinal number I can correlate an even number.
>
> But suppose I say, "Well go on—correlate them." Is

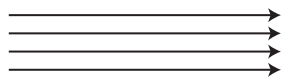
it at once clear what I mean? Is there only one technique for correlating cardinal and even numbers?

You can interpret "correlate" in such a way that you'll say, "Yes, there are as many . . ." But in what sense can you say you have *proved* this? You do a new thing and you *call* it "correlating them one-one"; and you call an entirely new thing "having the same number". All right. But you have not found two classes which have the same number; you have only invented a new way of looking at the thing. (*LFM*, pp. 160–61)

A parallel discussion occurs in *Philosophical Investigations*, where Wittgenstein employs imagery that is strikingly similar to that used in depicting one-to-one correlations:

PI 86. Imagine a language-game . . . played with the help of a table. The signs given to B by A are now written ones. B has a table; in the first column are the signs used in the game, in the second pictures of building stones. A shews B such a written sign; B looks it up in the table, looks at the picture opposite, and so on. So the table is a rule which he follows in executing orders.—One learns to look the picture up in the table by receiving a training, and part of this training consists perhaps in the pupil's learning to pass with his finger horizontally from left to right; and so, as it were, to draw a series of horizontal lines on the table.

Suppose different ways of reading a table were now introduced; one time, as above, according to the schema:



another time like this:

or in some other way.—Such a schema is supplied with the table as the rule for its use.

Can we not now imagine further rules to explain *this* one? And, on the other hand, was the first table incomplete without the schema of arrows? Are the other tables incomplete without their schemata?

Well, isn't Wittgenstein just wrong? Doesn't the pairing procedure correlate the set of integers with the set of even numbers in the same way that men and women are correlated as dancing couples? Matching things up in this way is not, after all, an intellectually challenging achievement. What we fail to notice—and I think this is what Wittgenstein is trying to show us—is that the activity of pairing men and women as dancing couples, that way of making a one-to-one correlation, is deeply embedded in such practical activities as counting objects (and stopping at a certain point), recognizing that they are in contact with one another, keeping track of them in space and time, and so on. It is the mastery of these surrounding activities, the possession of these skills and habits, that provides the de facto solution to the paradox of interpretation with respect to pairing dancing couples. With respect to the one-to-one correspondences between the set of integers and the set of even numbers, virtually all of these connections with the initial case have been severed. To borrow a phrase from Wittgenstein's discussion of a private language, the *stage setting* that makes it seem natural to take one-to-one correspondence

as the criterion of sameness in size (or number) is gone, but, as we might put it, its memory lingers on.

Disrupting our complacency about applying the procedure of one-to-one correspondence to infinite sets, *if* Wittgenstein has succeeded in doing so, would interrupt the smooth flow of the story that seemed to take us almost effortlessly from dancing couples to a hierarchy of ever-increasing infinities. Isn't that too bad? Well yes, for people fond of exotic places.

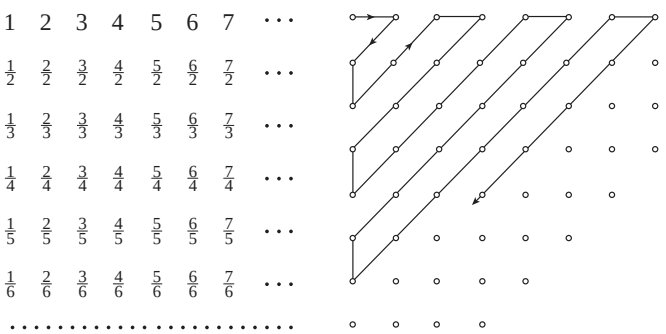
Appendix: CantorLand

Simple Pairing Procedure

1	2	3	4	5	.... n
↑	↑	↑	↑	↑	↑
↓	↓	↓	↓	↓	↓
2	4	6	8	10	....2n

Here the integers are put into one-to-one correspondence with the even integers, showing that these two sets have the same cardinality—which is usually taken to mean that they are the same size.

Serpentine Procedure



In what we will call the Serpentine Procedure, we set up an array in which every fraction will eventually occur. We then snake through it in the way indicated, writing down each fraction in turn. To avoid redundancy, fractions where the numerator and denominator have a common factor are deleted. In this way the rational numbers (i.e., all numbers expressible as fractions made up of integers) can be put into one-to-one correspondence with the integers. This shows that the set of integers and the set of rational numbers have the same cardinality—which is usually taken to mean that they are the same size.[13]

Diagonal Procedure

A Proposed Ordering of the Real Numbers

1	.**5** 7 2 9 4 3	
2	.9 **8** 7 6 3 2	
3	.2 3 **4** 7 6 8	
4	.7 6 3 **4** 4 3	
5	.4 3 2 2 **1** 5	
...	.....	
	Diagonal:	.58441.....
	Altered:	.11112.....

This argument is intended to show that the set of real numbers from 0 to 1 cannot be put into one-to-one correspondence with the set of integers. To see this, take the

[13] The diagram comes from Courant and Robbins, *What Is Mathematics?* (1996, p. 80). I have been guided by this wonderful book throughout this discussion of Cantor's proofs. The label "Serpentine Procedure" is my innovation.

list of numbers as the beginning of such an attempted one-to-one correspondence. However this correspondence is generated, we can show that there is a real number that will not appear in it. To see this, consider the diagonal number (given in boldface). We change each number on the diagonal to 1 unless it is 1, in which case we change it to 2. This number cannot appear in the list because the *n*th number in the *n*th line will always be wrong. Every attempt to give an ordered list of all the real numbers will fail in the same way. Hence, the real numbers cannot be put into one-to-one correspondence with the integers. Beyond this, since every integer has a matching real number, the set of real numbers must have a higher cardinality than the set of integers. This is usually taken to mean that the set of real numbers is larger than the set of integers.

The Power Set Argument

This is a proof that the cardinality of the power set of *S* is greater than the cardinality of the set *S*. The proof is trivial with respect to finite sets, so we will consider only infinite sets. As an example, let *S* be the set of all integers. Its power set, *S**, is the set of all the subsets of *S* including *S* itself and the empty set. We will assume, as the start of a reductio, that these two sets can be put into one-to-one correspondence and hence have the same cardinality. The integers in *S* can be related to their correlates in *S** in one of two ways: They can be members of the sets they are correlated with or not be members of their correlated sets. We can call the former Insiders, the latter Outsiders. For example, if 2 is correlated with {2,3,4}, then 2 is an Insider. If it is correlated with {1,3,4}, it is an Outsider.

We can now consider the set of all Outsiders in *S*. We will call it *O*. Since this is a subset of *S*, it will be an element in the power set *S**. Let's suppose that *s* is the element in *S* that has *O* as its correlate in the one-to-one mapping. By the definition of an Outsider, *s* is not in *O*. But by definition, if *s* is not in *O*, then it is an Outsider, hence it is in *O*. So *s* must both be contained in and not contained in *O*. This is a contradiction, so the assumption that the two sets can be put into one-to-one correspondence is false. Since every integer in *S* has a corresponding unit set containing it in *S** (for example, 2 paired with {2}), it is clear that *S** has a cardinality at least as high as the cardinality of *S*. Since, as has been shown, their cardinalities cannot be the same, *S** must have a higher cardinality than *S*. This argument can be repeated endlessly, producing an endless series of ever-greater infinite sets.

Chapter Five

Wittgenstein on Logical Consistency

"Contradiction destroys the calculus"—what gives it this special position? With a little imagination, I believe, it can certainly be shaken. (*RFM* VII 15)

But what if beings were . . . found whose laws of thought directly contradicted our own and therefore frequently led to contrary results in practice as well? The psychological logician could only simply acknowledge this and say: those laws are valid for them, these for us. I would say: here we have a hitherto unknown kind of madness. (Gottlob Frege, *The Basic Laws of Arithmetic*, vol. 1, p. xvi)

Wittgenstein has a reputation for having a laissez-faire attitude toward inconsistencies, paradoxes, contradictions, and the like. He doesn't. As the passage given above indicates, his reflections on these notions have a specific target: the idea that the occurrence of a contradiction in a calculus (for example, a logical system) destroys it. Wittgenstein's central claim is that this, as a *matter of fact*, is false. Wittgenstein's fullest defense of this claim and related matters occurs in the lectures on the foundations of mathematics that he presented at Cambridge in 1939.

Three Controversial Claims

In these lectures, Wittgenstein presents examples that illustrate the following claims:

1. A system containing a hidden (that is, as yet undiscovered) inconsistency may be perfectly all right (in Wittgenstein's words, "good as gold").
2. Even after an inconsistency is discovered in a system, it is all right to allow it to stand uncorrected because it can do no harm.
3. Finally, if we wish to remove the inconsistency, this will, in any case, prove an easy task.

Claim 1

If by a hidden contradiction one means a contradiction that hasn't been noticed, then, according to Wittgenstein, "as long as it is hidden . . . it is as good as gold" (*LFM*, p. 219). Consider a case in point: In 1903, on the eve of the publication of the second volume of his *Die Grundgesetz der Arithmetik*, Gottlob Frege received a letter from Bertrand Russell informing him of an inconsistency in the logical system presented in the first volume of this work. Is Wittgenstein committed to saying that Frege's system "was as good as gold" up to the time that Russell found an inconsistency in it? Yes, I think he is saying just this.

This seeming tolerance for inconsistency would, as Wittgenstein knew, strike most logicians as outrageous. On what we might call the standard view, a system containing a contradiction will have at least the following

"flaws":[1] If a system contains a contradiction, then it contains a falsehood—indeed a necessary falsehood. If a system contains a contradiction, it is possible, in that system, to infer something false from something true, thus violating the demand that logical inferences be truth-preserving. Finally, if a system contains a contradiction, it is possible, using standard logical procedures, to derive any proposition whatsoever, whether true or false, in that system.

How, then, could a system with the flaws listed above be, in Wittgenstein's words, good as gold, at least as long as they remain out of sight? Couldn't the contradiction, though unnoticed, lead to practical difficulties? Turing raised just this question:

> [Suppose] you have a logical system, a system of calculations, which you use in order to build bridges. You give the system to your clerks and they build a bridge with it and the bridge falls down. You then find a contradiction in the system. (*LFM*, p. 212)

Charles Chihara, spelling out this criticism, puts the matter this way:

> It is not hard to see how, by relying on such a system in reasoning about, say, the number of steel beams of such and such tested strength needed in a bridge to support a load of *N* tons, a disaster could result. Here, we could imagine the engineers working with such a

[1] I use the expression "standard view" to acknowledge the existence of nonclassical logical systems that, in one way or another, tolerate or accommodate contradictions—so-called paraconsistent logics, for example.

> large number of premises and carrying out such an intricate chain of inferences that a computer is used to check their work. If we imagine that they carry out their inferences rather mechanically, following set routines in accordance with general rules of strategy (as many students in logic courses do), it is not hard to see how they could start with true premises and end with false conclusions without noticing anything wrong with their logical system. (Chihara 1977, pp. 377–78)[2]

Let us suppose, as Chihara suggests, that the bridge collapses because employing the system yielded false values for stress levels. Doesn't this show that it is at least possible for an inconsistent logic to get us into trouble even when the inconsistency is out of sight? Wittgenstein refuses to acknowledge this.

> Now it does not sound quite right to say that a bridge might fall down because of a contradiction. We have an idea of the sort of mistake which would lead to a bridge falling.
>
> (a) We've got hold of a wrong natural law—a wrong coefficient.
>
> (b) There has been a mistake in calculation—someone has multiplied wrongly.
>
> The first case obviously has nothing to do with a contradiction; and the second is not quite clear. (*LFM*, p. 211)

[2] In this discussion I will, in various places, rely on Charles Chihara's excellent article, "Wittgenstein's Analysis of the Paradoxes in His Lectures on the Foundations of Mathematics" (Chihara 1977).

Now in a way Wittgenstein is right: It *doesn't* sound quite right to say that material errors can be due to the logical system employed, for logic, being empirically neutral, cannot determine anything about stress levels one way or another. That, I think, is what he is trying to get at. The difficulty with this response is this: A *consistent* logical system cannot generate an empirical falsehood, but an inconsistent one can.

Chihara summarizes the situation as follows:

> Suppose that the bridge the engineers design in our example subsequently collapses. Surely, we could distinguish at least three different possible explanations for the disaster: (1) the empirical theories and data the engineers relied upon were inaccurate or incorrect; (2) they made mistakes in calculation or didn't follow their rules of derivation correctly; (3) the logical system they used was unsound and led them to make invalid inferences (that is, they followed the rules of derivation correctly, but their calculus was wrong). (Chihara 1977, pp. 378–79)

Wittgenstein seems to be denying this third possibility without providing any satisfactory reason for doing so.

Can anything be said in defense of Wittgenstein's treatment of hidden contradictions? Wittgenstein could—and indeed does—object to the notion (or image) of a contradiction being hidden in a logical system like a germ hidden in the body. On his view, Russell's paradox was not lurking in Frege's system from the time that he formulated it. It was not there to be discovered; instead, it is something that *arises* in Frege's system when its rules are implemented in a particular way. I think that may be right: The germ imagery can be misleading and, perhaps,

can lead us into philosophical confusion. I cannot, however, see how this point is of use in the present case. Turing, and Chihara following him, have argued that a system of logic whose rules are not truth-preserving can get us into material trouble, and this is true whether this aspect of the system is hidden or not. I do not see how anything Wittgenstein says is responsive to this criticism.

Wittgenstein's contentious claim is that a system containing a hidden contradiction *is* as good as gold. With more modesty he could have said that a system containing a hidden contradiction *could be* as good as gold. In fact, at one place his wording shows a drift in that direction. To his own question "Why are people afraid of contradictions?" Wittgenstein responds:

> Turing says, "Because something may go wrong with the applications." But nothing *need* go wrong. (*LFM*, p. 217, emphasis added)

I think there is much to be said for the weaker claim that nothing *need* go wrong when one employs a system containing a hidden contradiction. This is important, and I will come back to it.

Claim 2

Beyond saying that a system containing a hidden contradiction is as good as gold as long as it goes unnoticed, Wittgenstein further says that "when it comes out in the open it can do no harm" (*LFM*, p. 219). As already noted, one of the harms that arises from employing a system that contains a contradiction is that in such a system it is possible to derive any proposition whatsoever, including

falsehood upon falsehood. Here, in a rather casual way, is how the proof runs:

1. *p and not-p*	Assumption
2. *p*	From 1
3. *p or q*	From 2, by *or* introduction
4. *not-p*	From 1
5. *q*	From 3 and 4, disjunctive syllogism

We could call this the problem of *logical promiscuity*. I will call it the *spread problem* instead. There is a rich literature suggesting various formal ways of blocking this line of argument. I will not go into this matter here. Wittgenstein, as we shall see, will attempt to block it as well, but not by objecting to the formal validity of one of the steps in the proof.

The spread problem can be approached in a variety of ways. One way is to employ only provably consistent systems. That, of course, is just the restriction that Wittgenstein is calling into question. Another is to depart from standard/classical logic and develop a set of rules that does not yield the spread principle. Although some logicians who adopt this approach—advocates of so-called paraconsistent logics, for example—sometimes cite Wittgenstein as their inspiration, I do not think Wittgenstein would be attracted to such *technical* responses to the problem. Wittgenstein's way of dealing with spread problems is captured by the maxim "If in a derivation you encounter a contradiction, *stop right there and go no further*!" The difficulty with this approach appears in the following exchange between Wittgenstein and Turing:

> *Wittgenstein*: You might get *p. ~ p* by means of Frege's system. If you can draw any conclusion you like from

> it, then that, as far as I can see, is all the trouble you can get into. And I would say, "Well then, just don't draw any conclusions from a contradiction."
>
> *Turing*: But that would not be enough. For if one made that rule, one could get round it and get any conclusion which one liked without actually *going through* the contradiction [emphasis added].
>
> *Wittgenstein*: Well, we must continue this discussion next time. (*LFM*, p. 220)

In fact, Wittgenstein did not continue the discussion in the next lecture.

Turing's claim is that a system that allows one to establish a contradiction will also allow one to establish any proposition whatsoever without having to *go through* the contradiction. The formal proof of this is somewhat complicated. I will not present it here.[3] The following simple example will illustrate how this can happen. If one adds axiom schema *p or q* to the standard axiom schemata of propositional logic, the system then becomes inconsistent. This is easily shown:

1. *p or q* Added (bad) axiom

Then, substituting *p and not-p* for both *p* and *q*, we get:

2. *(p and not-p) or (p and not-p)*

From which we can infer:

3. *p and not-p*

We can, as shown above, now use this derived contradiction to prove any arbitrary proposition we please, say, *q*. We can—and this will illustrate the point Turing is mak-

[3] Chihara provides one (Chihara 1977, p. 377).

ing—also use the (bad) axiom *p or q* to prove any arbitrary proposition, *r*; *without passing through* the contradiction:

1. *p or q*	Added (bad) axiom
2. *r or r*	Substituting *r* for both *p* and *q* (Yes, this is okay.)
3. *r*	

To those unfamiliar with modern logic, these maneuvers may seem arcane and artificial. Given time, it would be possible to show what motivates them. The point is that Turing's criticism turns on simple points of modern logic, and though hard to believe, it does seem, as Chihara suggests, that Wittgenstein doesn't understand it. At the close of Lecture 22 Wittgenstein seems to promise to return to this topic in his next lecture, but he doesn't. Furthermore, as Chihara points out, Wittgenstein repeats his maxim—stop when you encounter a contradiction—twice more later in his lectures (*LFM*, pp. 227 and 230), seemingly oblivious to the force of the objection that Turing has brought against it.

Claim 3

Wittgenstein's third suggestion takes the following form: Don't worry about contradictions until one shows up, and when one does, just fix the rules of the system in a way that blocks the derivation of a contradiction. This seems quite reasonable, but Wittgenstein, in effect, further suggests that this will be easy enough to do. To illustrate how easy this can be, Wittgenstein cites Russell's invention of type theory as a way of avoiding the paradox

found in Frege's system. Chihara demurs. He points out that Wittgenstein mischaracterizes the paradox that Russell found in Frege's system and further shows no understanding of the complexities involved in developing a type-theoretic solution to it. Chihara concludes, and I think correctly, that "Wittgenstein's conception of Russell's solution to the paradoxes is both superficial and erroneous" (Chihara 1977, p. 376).

So far, things have not been going well in my effort to provide a sympathetic interpretation of Wittgenstein's views concerning contradictions. I have conceded that Turing and Chihara are right in saying that part of Wittgenstein's treatment of contradictions involves an elementary logical misunderstanding. Now instead of acknowledging this, it may seem incumbent on a charitable interpreter to provide a reading of the text that avoids such an attribution. In general this is a proper demand, but there are times when the most charitable interpretation of a text involves acknowledging serious shortcomings in order to bracket them and clear the way for an examination of matters of more central importance.

Wittgenstein's Central Considerations

Chihara is right, I believe, in saying that Wittgenstein does not offer a satisfactory response to Turing's criticism of the *stop-right-there* method of dealing with what I have called the spread problem. He may even be right in suggesting that Wittgenstein missed its point. This, however, is compatible with the possibility that Wittgenstein has at his disposal *other* reasons for thinking that a system may be useful even though it is formally inconsistent. The key move here is the modest response, already

noted, that Wittgenstein makes to one of Turing's criticisms:

> Turing says, "Because something may go wrong with the applications." But nothing *need* go wrong. (*LFM*, p. 217, emphasis added)

To this end, Wittgenstein offers a series of examples of systems of rules that are formally inconsistent or dilemma-prone but yet, for all that, serviceable in their employment.

He begins by comparing a contradiction in a logical system with one found in the statutes of a particular country.

> Suppose that there is a contradiction in the statutes of a particular country. There might be a statute that on feast days the vice-president had to sit next to the president, and another statute that he had to sit between two ladies. This contradiction may remain unnoticed for some time, if he is constantly ill on feast-days. But one day a feast comes and he is not ill. Then what do we do? I may say, "We must get rid of this contradiction." All right, but does that vitiate what we did before? Not at all.
>
> Or suppose that we always acted according to the first rule: he is always put next to the president, and we never notice the other rule. That is all right: the contradiction does not do any harm.[4] (*LFM*, p. 210)

Here is one of his more bizarre examples:

> Suppose people had built a prison, and that the point of it is to keep the prisoners apart. Each prisoner can

[4] Wittgenstein does not consider the possibility that the president might be a woman, a fact pointed out to me by my wife.

> move along certain corridors and into certain rooms; but the rooms and corridors are so arranged that no two prisoners can ever meet.
>
> We could imagine that the system of corridors is very complicated—so that you might not notice that one of the prisoners can after all get by a rather complicated route into the room of another prisoner. So you have forfeited the point of this arrangement.
>
> Now suppose first that none of the prisoners ever noticed this possibility, and that none of them ever went that way. We could imagine that whenever two corridors cross at right angles, they always go straight on and never think of turning the corner. And suppose that the builder himself had never been struck by the possibility of their turning the corner at a crossing. And so the prison functions as good as gold. (*LFM*, p. 221)

This passage follows a pattern that we have encountered before: Wittgenstein first says something that seems natural to say in the circumstances and then corrects himself, because what he has said embodies a confusion. In leaving a route open that would allow a prisoner to find his way to another prisoner's room, we have, he says, "forfeited the point of this arrangement." But have we? The next paragraph indicates maybe not. Since *in fact* prisoners in this jail never turn corners, they will never hit on or stumble into a route that could take them from their rooms to the rooms of other prisoners. This is a defactoist solution at a most primitive level. The point is that a set of rules need not fill all gaps and anticipate every contingency in order to be serviceable and, perhaps, even as good as gold.

Wittgenstein pursues this point in a number of places. Some of his examples are quite simple. The following

passage, cited earlier, occurs almost as a throwaway in a discussion of the demand that a sentence must have a definite sense:

> *PI* 99. If I say "I have locked the man up fast in the room—there is only one door left open"—then I simply haven't locked him in at all; his being locked in is a sham. One would be inclined to say here: "You haven't done anything at all". An enclosure with a hole in it is as good as *none*.—But is that true?

How are we to answer Wittgenstein's question? If we pause to think about it, it is not too hard to imagine cases where leaving one door open would *not* amount to doing nothing at all. Perhaps the building has many entrances and locking all but one would be enough to discourage would-be trespassers who, after trying a few doors and finding them locked, might give up and go away. This is not a perfect security system, but using it could be significantly more effective than doing nothing at all. Again, in a country where the common practice is to have shop doors open inward—as it is in most countries in Europe, but not in the United States—it might be reasonably safe to have one outward-opening door that is left unlocked. Finally, if the only unlocked door is in a secret entrance, then the system might, in fact, be as good as gold.

We can next examine a richer example of a "flawed" system being as good as gold despite the existence of a flaw.[5]

[5] I first presented the ideas developed in the following paragraphs at a conference in Bristol in 1974. The paper was later published (Fogelin 1976a) under the title "Hintikka's Game Theoretic Approach to Language." I have also offered an informal discussion of the notion that a system of rules can be inconsistent, yet still serve useful purposes, in the second chapter of *Walking the Tightrope of Reason* (Fogelin 2003).

> *RFM* VII 29. I am defining a game and I say: "If you move like *this*, then I move like this, and if you do that, then I do *this*.—Now play." And now he makes a move, or something that I have to accept as a move and when I want to reply according to my rules, whatever I do proves to conflict with the rules.

To see the force of this remark, suppose there is a game—I will call it *Ludwig*—with the richness and complexity of chess. It has been played for centuries and the rules have been carefully codified. Books and magazines are dedicated to it; world championships are held; and so on. We can imagine two novices who are beginning to learn the game by simply practicing legal moves. In the process they wind up in a position where, according to the rules, one of the players is required to make a move but also forbidden to do so. Since a series of legal moves can yield this result, we will say that Ludwig is *dilemma prone*.[6]

This example may seem wholly artificial, for how, during the centuries of expert play, could such a flaw in the rules go unnoticed? It could happen this way: Though the novices moved their pieces in legal ways, they had not yet learned the *point* of the game. In *Remarks on the Foundations of Mathematics*, Wittgenstein cites the notion of the *point* of the game in order to draw a distinction between essential and nonessential rules.

> *RFM* I, Appendix 1, 19. Let us say: the meaning of a piece (a figure) is its role in the game.—Now before

[6] In Wittgenstein's example, one of the players winds up in a position where there are no legal moves open to him. When this happens in chess, the game is declared a draw. In the game of Ludwig, a player finds himself in a position where he is under the constraints of incompatible demands. This strikes me as being a more apt analogy with the case of a system containing a hidden contradiction than Wittgenstein's own example, but the spirit is the same.

> the start of any chess-game let it be decided by lot which of the players gets white. For this purpose one player holds a king in each closed hand and the other chooses one of the hands at random. Will it be reckoned as part of the role of the king in chess that it is used for drawing by lot?
>
> *RFM* I, Appendix 1, 20. Thus even in a game I am inclined to distinguish between essential and inessential. The game, I should like to say, does not just have rules; it has a point.

We can also use the notion of the point of a game in order to distinguish playing the game from merely practicing moves. In playing the game there are two different constraints at work: those that involve the legality of moves and those that are guided by the outcome a person is trying to achieve by making moves—to win, or at least not to lose. The first kind of constraints would be found in rule-books for Ludwig; the second kind would be presented in books with titles like *Learn to Play Ludwig Like a Master*. In our example, the two novices understand the difference between a legal and an illegal move, but not the difference between a smart move and a dumb move. Thus, by producing a sequence of dumb—or at least unmotivated—moves, they can stumble onto an incoherence in the system of rules that no one who understands what the game is all about would ever encounter.

The game of Ludwig provides an example of the way in which a practice can be immune to—or inoculated against—internal incoherence that is richer than the previous examples we have examined. The prison example depended on the fact that the prisoners moved zombie-like, never thinking of making turns at intersecting corridors. The incoherence in the seating arrangement for

the town's feast-day meetings never surfaced just because the members of the committee whose attendance would reveal the incoherence simply never showed up together. In the Ludwig example, the expert players are not insulated from dilemma by ignorance, incompetence, or chance. As the situation is described, the expert players know nothing of the dilemma but are protected from it through being engaged in an activity that in fact never takes them into the region where the dilemma can arise. It is important to see that, in daily life, we are almost always protected from paradoxes and dilemmas in the same way.

In all three examples—the seating arrangement, the prison, and Ludwig—it is a de facto feature of the situation that allows a system to work despite what we might call a formal incoherence. There does, however, seem to be an important difference between the first two examples and the game of Ludwig: The expert players, unlike the novices, are constrained in the moves they make, not only by the formal rules of the game but also by pragmatic considerations, some of which may also take the form of rules. There are certain things that one should never do although it would not be illegal to do them. In a serious game of chess, one should not expose one's king to immediate checkmate. These pragmatic rules vary in importance and in particularity. For an experienced player, many of them become second nature. Given these dual constraints—the formal and the pragmatic—the following situation is possible: A system whose formal rules are dilemma-prone can be pragmatically shielded from dilemma. We might say that such a system, though not formally consistent, is pragmatically consistent, or, for short, pragma-consistent. Notice that I

am not speaking about a pragma-consistent *logic*, that is, something akin to so-called paraconsistent logics. Pragmatic constraints do nothing to remove the dilemma-proneness of a formal system, for they do not alter the structure of the formal system in any way.[7]

Earlier I attributed three claims to Wittgenstein that many of his critics, Chihara among them, have found outrageous.

1. A system containing a hidden (i.e., as yet undiscovered) inconsistency may be perfectly all right.

2. Even after an inconsistency is discovered in a system, it is all right to allow it to stand uncorrected because it can do no harm.

3. Finally, if we wish to remove the inconsistency, this will prove an easy task.

How plausible are these claims when applied to the game of Ludwig as I have described it?

1. Was Ludwig "good as gold" over the long period in which its dilemma went unnoticed? Perhaps Wittgenstein's expression "good as gold" is needlessly hyperbolic, but at least this much is true: The dilemma-proneness causes no difficulties in serious play, nor is it ever likely to. Thus, if we treat the game of Ludwig as a rich activity that human beings engage in, then the inherent dilemma-proneness of the formal rules may strike us as irrelevant. Taken this

[7] In *Walking the Tightrope of Reason*, I referred to systems of rules that have this feature as being "Ludwigean" (Fogelin 2003, p. 47). "Pragma-consistent," with its specious technical ring, sounds more dignified.

way, the discovery of such an inconsistency would not jeopardize the legitimacy of past play in any way. I think that everyone except possibly a die-hard logical purist would agree with this.

2. *What happens after the inconsistency in the rules of Ludwig is discovered?* From what has already been said, it should be obvious that the International Ludwig Association, apprised of the dilemma-proneness of Ludwig, might decide to let its rules stand just as they are. The dilemma-proneness of Ludwig has caused no difficulties in the past and there seems no likelihood that it will cause difficulties in the future. Its dilemma-proneness does not, by itself, provide an adequate reason for abandoning (or even modifying) the game of Ludwig. In a striking passage, Wittgenstein indicates that a parallel situation could obtain if an inconsistency were found in arithmetic.

> *RFM* VII 35. What sort of certainty is it that is based on the fact that in general there *won't* actually be a run on the banks by all their customers; though they would break if it did happen?! Well, it is a *different* kind of certainty from the more primitive one, but it is a kind of certainty all the same.
>
> I mean: if a contradiction were now actually found in arithmetic—that would only prove that an arithmetic with *such* a contradiction in it could render very good service; and it will be better for us to modify our concept of the certainty required, than to say that it would really not yet have been a proper arithmetic.

3. *Finally, Wittgenstein holds that a system that contains an inconsistency in need of correction will cause no long-term problems, because the system can be easily fixed in a way that avoids the inconsistency.* This is a more dubious claim. Re-

call that Wittgenstein cited Russell's development of the theory of types as an example of how easy it is to alter rules to avoid a paradox when one occurs. Chihara rightly points out that developing a way to avoid the paradox that Russell found in Frege's logical system was not a routine matter. The same could be true for making the rules of Ludwig dilemma-proof. Though the dilemma never makes an appearance in serious play, efforts to avoid it could have an adverse impact on the game. Perhaps avoiding the dilemma would make the rules unacceptably complex. It might trivialize that game. Games might become endlessly long or uninterestingly short. And so on.

Application

To turn to a related but somewhat different subject, an important aspect of Wittgenstein's treatment of logical paradoxes is his claim that they arise in contexts where a concern with application is pushed aside. Thus, with respect to Russell's paradox, he tells us:

> *RFM* VII 10. What Russell's '$\sim f(f)$' lacks above all is application, and hence meaning.
>
> If we do apply this form, however, that is not to say that '$f(f)$' need be a proposition in any ordinary sense or '$f(\zeta)$' a propositional function. For the concept of a proposition, apart from that of a proposition of logic, is only explained in Russell in its general conventional features.
>
> Here one is looking at language without looking at the language game.

The response to this is, I suppose, that the notation for the Russell property (or the notation for the correspond-

ing Russell class) has perfectly clear applications. For example, something is a Russell class just in case it is a class that is not a member of itself. That's how the expression "a Russell class" is applied. Wittgenstein's response to this is that here "one is looking at language without looking at the language-game." Clearly, the kind of language game Wittgenstein has in mind is one that uses a symbolism that has an employment outside mathematics. What useful or practical purposes are served by introducing this symbolism? That's Wittgenstein's question. Wittgenstein makes this point with reference to the Grelling paradox.[8]

> *RFM* VII 28. Why shouldn't it be said that such a contradiction as: 'heterological' ∈ heterological ⇔ ~ ('heterological' heterological), shews a logical property of the concept 'heterological'?

It is not immediately clear how he expects us to answer this question. We need context lenses to get it right. The passage continues:

> " 'Two-syllabled' is heterological", or " 'Four-syllabled' is not heterological" are empirical propositions. It might be important in some contexts to find out whether adjectives possess the properties they stand for or not. The word "heterological" would in that case be used in a language-game. But now, is the proposition " '*h*' ∈ *h*" supposed to be an empirical proposition? It obviously is not one, nor should we admit it as a proposition in our language-game even if we had not discovered the contradiction.

[8] At *PR* XI 122, Wittgenstein attributes what he calls the "heterological" paradox to Hermann Weyl.

Wittgenstein first points out that the adjective "heterological" can have unproblematic applications outside of logic and mathematics. In saying that "two-syllabled" is heterological, Wittgenstein is, in effect, employing the following disquotational schema:

"ϕ" is heterological $\Leftrightarrow$ "ϕ" is not ϕ.

Thus, through substitution, we can get:

(1) "Two-syllabled" is heterological $\Leftrightarrow$ "two-syllabled" is not two-syllabled.

Since, *as a matter of fact*, "two-syllabled" is not two-syllabled, "two-syllabled" is heterological. (1) is thus a true empirical proposition implying the adjective "heterological." No problems there. The situation is, however, fundamentally different with respect to the following instantiation of the disquotational schema.

(2) "Heterological" is heterological $\Leftrightarrow$ "heterological" is not heterological.

Ascribing the property of being heterological to "heterological" does not ascribe any empirical feature, or for that matter, any sort of external feature that can be appealed to to determine whether "heterological" is heterological or not. It is easy to see why not. In (1), "heterological" finds its application by way of the quality ϕ that emerges from the disquotation of "ϕ." In (2), no such pathway to application is provided.

We can put Wittgenstein's criticism of Russell this way: Not questioning the meaningfulness of (2) or the meaningfulness of other paradox-generating sentences, Russell plunges straight ahead to derive a contradiction. But why doesn't Russell recognize that in (2)—and in other

paradox-generating sentences—words are being used in ways that disconnect them from their *application*? Wittgenstein's answer is given in a passage already cited: "one is looking at language without looking at the language-game" (*RFM* VII 10).

What then should we say? Does "'heterological' ∈ heterological ⇔ ~('heterological' ∈ heterological)," show a logical property of the concept 'heterological' or not? Here is what Wittgenstein thinks it does show.

> *RFM* VII 28. '*h*' ∈ *h* ⇔ ~ ('*h*' ∈ *h*) might be called 'a true contradiction'.—But this contradiction is not a significant proposition! Agreed, but the tautologies of logic aren't either.
>
> "The contradiction is true" means: it is proved; derived from the rules for the word "*h*". Its employment is, to shew that "'*h*'" is one of those words which do not yield a proposition when inserted into 'ζ ∈ *h*'.
>
> "The contradiction is true" means: this really is a contradiction, and so you cannot use the word "'*h*'" as an argument in 'ζ ∈ *h*'.

What the Grelling paradox shows, but of course does not state, is that the adjective with a useful employment can lose its employment when it is turned back on itself.

The Liar Paradox

I will finish this discussion of paradoxes by examining Wittgenstein's reflections on the Liar paradox. Wittgenstein's treatment of this paradox parallels his treatment of the Heterological paradox, but in some ways it is inter-

estingly different. The following passage comes from Lecture 20 of *Lectures on the Foundations of Mathematics*:

> Think of the case of the Liar. It is very queer in a way that this should have puzzled anyone—much more extraordinary than you might think: that this should be the thing to worry human beings. Because the thing works like this: if a man says "I am lying" we say that it follows that he is not lying, from which it follows that he is lying and so on. Well, so what? . . . It doesn't matter. . . .
>
> Now suppose a man says "I am lying" and I say "Therefore you are not, therefore you are, therefore you are not . . ." —What is wrong? Nothing. Except that it is of no use; it is just a useless language-game, and why should anybody be excited?
>
> . . . .
>
> *Turing*: What puzzles one is that one usually uses a contradiction as a criterion for having done something wrong. But in this case one cannot find anything done wrong.
>
> *Wittgenstein*: Yes—and more: nothing has been done wrong. One may say, "This can only be explained by a theory of types." But what is there which needs to be explained? (*LFM*, pp. 206–7)

An important feature of this exchange between Wittgenstein and Turing is Wittgenstein's insistence that there is nothing puzzling—nothing to be wondered at—concerning the appearance of the Liar paradox. As he sees it, everything is completely out in the open. There is nothing in need of explanation. We start with expressions that have unproblematic applications, for example, "is

false" or "is heterological." We then employ them in a manner that undercuts the application that gave them sense. The harm, if there is any, has already been done. We then derive an expression with the *form* of a contradiction. Now in speaking about an expression with the form of a contradiction, I may be going beyond what Wittgenstein actually says, but I think that it is an apt way of making his point: A substitution into the schema $p \,\&\, \sim p$ will count as a contradiction only if what is substituted for p has a truth value. Neither the Liar Sentence nor the Grelling Sentence satisfy this condition.

Given this reading, what are we to make of the closing exchange between Wittgenstein and Turing, where Wittgenstein says that in generating the Liar paradox "nothing has been done wrong"? This seemingly outrageous claim is grounded in something he says a few sentences earlier:

> If we have a use of "I'm lying" from which it follows "I'm not lying"—isn't this just a useless game? (*LFM*, p. 207)

I take it that we are supposed to say yes, it would be a useless game. But suppose that someone—recognizing its uselessness—chose to play this game, would he have done anything wrong in the way he plays it? Wittgenstein's explicit answer would be no, he played the game just fine. Of course, if he came to recognize the uselessness of the game, he might stop playing it and games like it. But maybe not.[9]

[9] In *Zettel* Wittgenstein presents an alternative way of dealing with the Liar paradox:

> *Z* 691. "The Cretan Liar". He might have written "This proposition is false" instead of "I am lying". The answer

Wittgenstein's treatment of the Liar paradox leaves Chihara sputtering with indignation:

> What is so puzzling about The Liar is the fact that the argument is so simple and tight. Putting one's finger on the wrong move is extremely difficult, as practically everyone who has thought deeply about the paradox will agree. After all, The Liar has remained a paradox without a generally accepted solution for approximately two thousand years, despite strenuous attempts at solving it by some of the very best minds in logic and philosophy. Wittgenstein's hasty dismissal of this ancient and venerable problem is, in my opinion,

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> would be: "Very well, but which proposition do you mean?"—"Well, *this* proposition".—"I understand, but which is the proposition mentioned in it?"—"*This* one"—"Good, and which proposition does *it* refer to?" and so on. Thus he would be unable to explain what he means until he passes to a complete proposition.—We may also say: The fundamental error lies in one's thinking that a phrase e.g. "This proposition" can as it were allude to its object (point to it from far off) without having to go proxy for it.

Looked at this way, we never reach a terminating statement (or any statement at all) whose truth can be evaluated. All we get is a useless repetition of the same question and answer over and over again. No paradox emerges, but we have put words together in a way that puts us on an endless treadmill.

Roy Sorensen reminded me that Gilbert Ryle adopted a similar response to the Liar paradox:

> If unpacked, our pretended assertion ['The current statement is false'] would run 'The current statement {namely, that the current statement [namely that the current statement (namely that the current statement . . .'. The brackets are never closed; no verb is ever reached; no statement of which we can even ask whether it is true or false is ever adduced. (Ryle 1951, p. 68)

> neither well-reasoned nor insightful. (Chihara 1977, pp. 379–80)

There is another way of looking at this. That more than two thousand years of strenuous effort by the best minds in logic and philosophy have yet to yield a generally accepted solution to the Liar paradox creates a strong presumption that the formulation of this paradox embodies a conceptual misunderstanding—a conceptual misunderstanding that arises naturally, and hence is hard to shake.

Frege's "Hitherto Unknown Kind of Madness"

At the head of this chapter I cited a passage from Frege where he considers beings "whose laws of thought directly contradicted our own and therefore frequently led to contrary results in practice as well." According to Frege, the psychological logician could only say: "Those laws are valid for them, these for us." For his part, Frege would say, "Here we have a hitherto unknown kind of madness." What would Wittgenstein say?

There is, I think, a temptation to lump Wittgenstein in with what Frege calls the *psychological logicians* and then either defend or reject his position on the basis of this reading. I think this is a mistake. I think the first thing Wittgenstein would say is that Frege's example is underdescribed. We want to know how we can tell that these creatures follow laws that contradict our own. Do they give nodding assent to an explicit contradiction? If so, how can we be sure that they understand the sentence as we do? Just as important, how do their laws of thought lead to "contrary results in practice as well"? Contrary to what? Contrary to the way we act, or (somehow) inher-

ently contrary? Perhaps, using their logical procedures, they build bridges that invariably fall down and construct boilers that always blow up.

Confronted with such beings, is Wittgenstein forced to say, in Frege's words, "Those laws are valid for them, though not for us?" There are, I suppose, people who would hold that saying anything else amounts to cultural chauvinism. This, however, is not what Wittgenstein says. In *PI* 207 he considers a tribe populated with creatures similar to Frege's beings.

> *PI* 207. Let us imagine that the people in that country carried on the usual human activities and in the course of them employed, apparently, an articulate language. If we watch their behaviour we find it intelligible, it seems 'logical'. But when we try to learn their language we find it impossible to do so. For there is no regular connexion between what they say, the sounds they make, and their actions; but still these sounds are not superfluous, for if we gag one of the people, it has the same consequences as with us; without the sounds their actions fall into confusion—as I feel like putting it.
>
> Are we to say that these people have a language: orders, reports, and the rest?
>
> There is not enough regularity for us to call it "language".

Wittgenstein adopts the same strategy in the following striking passage:

> *PI* 342.William James, in order to shew that thought is possible without speech, quotes the recollection of a deaf-mute, Mr. Ballard, who wrote that in his early youth, even before he could speak, he had had thoughts

> about God and the world.—What can he have meant?—Ballard writes: "It was during those delightful rides, some two or three years before my initiation into the rudiments of written language, that I began to ask myself the question: how came the world into being?"—Are you sure—one would like to ask—that this is the correct translation of your wordless thought into words? And why does this question—which otherwise seems not to exist—raise its head here? Do I want to say that the writer's memory deceives him?—I don't even know if I should say *that*. These recollections are a queer memory phenomenon,—and I do not know what conclusions one can draw from them about the past of the man who recounts them.[10]

Wittgenstein, I think, would deal with Frege's beings in just the same way. He would not say that these people have linguistic practices valid for them. He would not declare their practices insane. He would, instead, refuse to take a stand on a matter too indeterminate to admit of an answer.

[10] This is another example of a passage with the paradox of interpretation embedded in it.

Coda

When it looks as if___, we should look out. (*RFM* II 41)

This work offers a reading of Wittgenstein's later philosophy that emerges from taking his methodological statements at face value. Taking him at his word has, to a large extent, involved letting him speak for himself. This approach involves stressing certain passages at the expense of others, and that, of course, itself imposes a strong interpretation on the text. I have tried to mitigate this problem by concentrating on persistent themes in Wittgenstein's later philosophy—themes that were already falling into place in the early 1930s, for example, in the material found in *Philosophical Grammar*. To borrow one of Wittgenstein's expressions, I have concentrated on aspects that give his later reflections their *characteristic physiognomy*.

One thing I have *not* tried to show is that the insights found in Wittgenstein's later philosophy can be integrated with some of the standard forms of philosophizing that flourish on the contemporary scene. I have in mind, among other things, the reliance on what are called *intuitions* as desiderata (or at least prima facie desiderata) governing conceptual investigations. There are obvious

problems with appeals to intuition that are generally acknowledged. People disagree in their intuitions, or at least seem to. They can also differ in the priorities they assign to intuitions. In crucial cases, intuitions often flicker or go out. To devotees of intuition, these supposed difficulties merely show that properly invoking intuitions is a subtle and sophisticated enterprise not to be left to amateurs. Fine distinctions must be drawn and then modified in the light of further intuitions. With a great deal of carefully circumscribed effort, it may be possible to tease out an intuitively based response to a philosophical problem that at least approximates what is called reflective equilibrium.

Wittgenstein's approach cannot be used to further such an enterprise because it stands directly opposed to it. What we are inclined to say—even strongly inclined to say—when presented with a philosophical question is not, for Wittgenstein, part of the *data set* to be used to answer the question. It should, instead, be viewed with suspicion. We should step back and investigate how the question arose and why, in the context of philosophical reflection, we are inclined to answer it in the particular way we do. This is a sermon Wittgenstein preaches over and over again. Passages to this effect have been cited already. Here are two more:

> The solipsist who say "only I feel real pain", "only I really see (or hear)" is not stating an opinion; and that's why he is so sure of what he says. He is irresistibly tempted to use a certain form of expression; but *we must yet find why he is*. (*B & B*, pp. 59–60, emphasis added)

> *PI* 299. Being unable—when we surrender ourselves to philosophical thought—to help saying such-and-such;

> being irresistibly inclined to say it—does not mean being forced into an assumption, or having an immediate perception or knowledge of a state of affairs.

For Wittgenstein, the inherent danger in responding to a philosophical question with what we are unhesitatingly inclined to say is that the response may inherit a conceptual confusion embodied in the question itself.

Wittgenstein, we have seen, illustrates this tendency to be captured by what we are spontaneously inclined to say by having an interlocutor respond with full conviction to a philosophical matter in a way that is completely at odds with daily beliefs, including the person's own daily beliefs. In a philosophical discussion one can find oneself inclined to say—even forced to say—"Only I can know whether I am really in pain; another person can only surmise it" *(PI* 246). Wittgenstein sometimes produces such spontaneous conceptual improprieties in his own voice, then corrects himself. When we philosophize, there seems to be an inherent tendency to start off on the wrong foot.

Let me return to an example of Wittgenstein's procedure cited earlier but not discussed in detail. It concerns Wittgenstein's brief—and to my mind, brilliant—treatment of personal identity. As we saw, in the *Blue and Brown Books* Wittgenstein presents the following example of a problem concerning personal identity:

> Imagine a man whose memories on the even days of his life comprise the events of all these days, skipping entirely what happened on the odd days. On the other hand, he remembers on an odd day what happened on previous odd days, but his memory then skips the even days without a feeling of discontinuity. If we like we

> can also assume that he has alternating appearances and characteristics on odd and even days.

Wittgenstein asks:

> Are we bound to say that here two persons are inhabiting the same body? That is, is it right to say that there are, and wrong to say that there aren't, or vice versa?

Many philosophers instinctively answer this question yes, we are forced to choose between one of these alternatives, and though the task may prove difficult, it is the philosopher's job to show which answer is correct. In characteristic fashion, Wittgenstein refuses to be saddled with this seemingly forced choice. Asking himself which choice he would make, he responds:

> Neither. For the *ordinary* use of the word "person" is what one might call a composite use suitable under the ordinary circumstances. If I assume, as I do, that these circumstances are changed, the application of the term "person" or "personality" has thereby changed; and if I wish to preserve this term and give it a use analogous to its former use, *I am at liberty to choose between many uses, that is, between many different kinds of analogy* [emphasis added]. One might say in such a case that the term "personality" hasn't got one legitimate heir only. (*B & B*, p. 62)

As Wittgenstein insists in a number of places, the use of expressions in our language depends on the existence of broad stabilities and continuities that we take for granted—and it is important that we *do* take them for granted. Without the stabilities and continuities in ourselves and in the world around us, certain language games

would, in fact, not arise. Ignoring this, contemporary philosophers often take some notion that has a "composite use suitable under the ordinary circumstances," and then ask whether it applies or does not apply in a radically nonstandard context. No ducking the question is allowed, when, in fact, ducking the question or, better, refusing to take the question head-on, is precisely the right thing to do.

Weird settings can generate skewed intuitions. Skewed intuitions, in turn, can destabilize common, unproblematic uses of language. Wittgenstein recognized all this and issued warnings concerning it. Largely, it seems, in vain.

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